Computational Economics **18:** 233–249, 2001.

A Computational Approach to the Fundamental Theorem of Asset Pricing in a Single-Period Market

F. ACEDO[1], F. BENITO[1], A. FALCÓ[2], A. RUBIA[1] and J. TORRES[1]

[1]*Departamento de Economía Financiera, Contabilidad y Marketing, Universidad de Alicante, Apdo. de Correos 99, 03080 Alicante, Spain;* [2]*Universidad Cardenal Herrera CEU, Comissari 1, 03203 Elche, Spain*

Abstract. We provide a new approach to the *Fundamental Theorem of Asset Pricing* based on the relation between the projection problem and equivalent least squares problem. More precisely, we use an iterative procedure in order to obtain solutions of a bounded least square problem. Under some conditions, this solution will give either the state price vector or the arbitrage opportunity of the problem under consideration.

Key words: asset pricing, arbitrage, mathematical finance

1991 Mathematics Subject Classification: primary 90A12; secondary 60F10

1. Introduction

The basic idea of the whole pricing philosophy in finance consists of the construction of a linear functional π which strictly separates the arbitrage opportunities obtained by trading strategies. Moreover, this functional provides the state price vector necessary for pricing contingent claims in a financial market. In a more formal way, if we denote by M the linear subspace of all trading strategies and by $\mathbb{K}$ the nonnegative cone, then there are no arbitrage opportunities if and only if $M \cap \mathbb{K} = \{\mathbf{0}\}$. This result is known as the *Fundamental Theorem of Asset Pricing*. We remark that the existence of state price vectors follows by the *Separating Hyperplane Theorem* (Rockafellar, 1990) – a version of the Hahn–Banach theorem (see Duffie, 1992). However, the Hahn–Banach theorem is not a useful tool to construct the state price vector. Despite its practical importance, a computational approach to the calibration of the state price vector has received little attention in the mathematical finance theory in discrete time (see Sheldon (1996), Theorem 1.2, Bingham and Kiesel (1998), Theorem 1.4.1, and Pliska (1997), 1.16). We remark that the single period model is important because it provides much of the intuition that is necessary for more general models of financial markets and because it is possible to combine the result of the single period in order to prove the *Fundamental*

Theorem of Asset Pricing for the Multi-Period Model (see Bingham and Kiesel (1998), Proposition 4.2.3 and Sheldon (1996), Section 2.3 and Theorem 3.1).

The aim of this paper is to give a constructive proof of this fundamental result. We comupte a linear measure, given by a vector, for which the Euclidean distance between the expected payment in the first period and the initial price vector in period zero is minimal. Indeed, this linear measure is the solution of a bounded linear square (BLS) problem. If the Euclidean distance is non-zero then the difference between these two vectors gives an arbitrage opportunity. Otherwise, if the Euclidean distance is zero, one of the following statements holds. If the linear measure is positive, this one is a state price vector and no arbitrage opportunities exist. If the linear measure has components, which depends on the final random payments and indices of these zero components, arbitrage opportunities exist or do not exist. In the first case we compute an arbitrage opportunity as the residual vector of a solution of a BLS problem. Otherwise, by using the linear measure, we construct a continuous path of state price vectors.

We note that our strategy is similar to the one used by Avellaneda (1998) for calibrating an asset-pricing model in continuous time (recall that we use a discrete time model). To this end Avellaneda gives an algorithm in order to obtain a risk-neutral probability that minimizes the Kullback–Leibler distance, also called the relative entropy, with respect to a given prior distribution. The main difference lies in the fact that in Avellaneda's result the existence of risk-neutral probabilities in the market is assumed. That is, he assumes the existence of state price vectors.

This paper is organized as follows. In the next section we introduce some definitions, we state the main result of this paper and give some numerical examples as its applications. In Section 3 we prove the main theorem of this paper.

2. Definition and Statement of Results

In this paper we consider a single period market, that is, we have two indices, namely $t = 0$ which is the current time, and $t = \Delta t$, which is the terminal date for all economic activities under consideration.

The financial market contains N traded financial assets, whose prices at time $t = 0$ are denoted by

$$\mathbf{S}_0 = [S_0^1 \; S_0^2 \ldots S_0^N]' \geq \mathbf{0},$$

here $'$ denotes the transpose of a matrix or vector. At time Δt, the owner of financial asset number i receives a random payment depending on the state of the world. We model this randomness by introducing a finite probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where $\Omega = \{\omega_1, \omega_2, \ldots, \omega_k\}$, $\mathcal{F} = \mathcal{P}(\Omega)$ and $\mathbb{P}(\omega_1) > 0$ for all $i \in \{1, 2, \ldots, k\}$. We note that the random payment arising from financial asset i is a $\mathbb{R}^k$-vector

$$[S_{\Delta t}^i(\omega_1), S_{\Delta t}^i(\omega_2), \ldots, S_{\Delta t}^i(\omega_k)]' \geq \mathbf{0}.$$

At time $t = 0$ the agents can buy and sell financial assets. The portfolio position of an individual agent is given by a trading strategy, which is a vector

$$\boldsymbol{\theta} = [\theta_1, \theta_2, \ldots, \theta_N]' \in \mathbb{R}^N.$$

Here θ_i denotes the quantity if the ith asset is bought at time $t = 0$, which may be negative if the agent has a short position, as well as positive if he has a long position.

The dynamics of this model using the trading strategy $\boldsymbol{\theta}$ are as follows:

1. At time $t = 0$ the agent invests the amount

$$\mathbf{S}_0'\boldsymbol{\theta} = \theta_1 S_0^1 + \theta_2 S_0^2 + \cdots + \theta_N S_0^N ,$$

2. and at time $t = \Delta t$ the agent receives a random payment $\mathbf{P}$ that we can represent by using a matrix as follows. Let

$$\mathbf{S}_{\Delta t} = \begin{bmatrix} S_{\Delta t}^1(\omega_1) & S_{\Delta t}^1(\omega_2) & \cdots & S_{\Delta t}^1(\omega_k) \\ S_{\Delta t}^2(\omega_1) & S_{\Delta t}^2(\omega_2) & \cdots & S_{\Delta t}^2(\omega_k) \\ \vdots & \vdots & \ddots & \vdots \\ S_{\Delta t}^N(\omega_1) & S_{\Delta t}^N(\omega_2) & \cdots & S_{\Delta t}^N(\omega_k) \end{bmatrix},$$

then

$$\mathbf{P} = \mathbf{S}_{\Delta t}'\boldsymbol{\theta} .$$

We remark that each component of vector $\mathbf{P}$ represents the payment received depending on the realized state of the world ω.

Then we can define an *arbitrage opportunity* as a vector $\boldsymbol{\theta} \in \mathbb{R}^N$ such that one of the following two conditions holds.

(**Arb1**): $\mathbf{S}_0'\boldsymbol{\theta} = 0$ and $\mathbf{P} = \mathbf{S}_{\Delta t}'\boldsymbol{\theta} \geq \mathbf{0}$, with $\mathbf{S}_{\Delta t}'\boldsymbol{\theta} \neq \mathbf{0}$.

(**Arb2**): $\mathbf{S}_0'\boldsymbol{\theta} < 0$ and $\mathbf{P} = \mathbf{S}_{\Delta t}'\boldsymbol{\theta} \geq \mathbf{0}$.

Note that in the case of an arbitrage opportunity which satisfies (Arb 1) the agent's net investment at time $t = 0$ is zero, and there exists a $\omega \in \Omega$ such that

$$\sum_{i=1}^{N} S_{\Delta t}^i(\omega)\theta_i > 0 ,$$

that is, there exists non-zero probability to obtain a 'free lunch'. In the case of condition (Arb 2), we have that $\mathbf{S}_0'\boldsymbol{\theta} < 0$, that is, the agent borrows money for consumption at time $t = 0$, and he does not have to repay anything at the time Δt.

By using the well-known result called the *Separating Hyperplane Theorem* (Rockafellar, 1990), which is a version of the *Hahn-Banach Theorem* (Sheldon, 1996) the following result follows (Bingham and Kiesel, 1998; Duffie, 1992; Pliska, 1997; Sheldon, 1996).

THEOREM 1. *There are no arbitrage opportunities if and only if there exists* $\Psi > \mathbf{0}$ *such that*

$$\mathbf{S}_{\Delta t}\Psi = \mathbf{S}_0 \,. \tag{1}$$

We will say that a vector $\Psi > \mathbf{0}$ satisfying (1) is a state price vector. Moreover, we can state that the *Separating Hyperplane Theorem implies the existence of a state price vector in the proof Theorem 1.*

The main goal of this paper is to construct either the state price vector if non-arbitrage opportunities exist or an arbitrage opportunity if there are no state price vectors. To see this we shall use an algorithm due to Dax (1993). More precisely, Dax's Algorithm provides a solution $\mathbf{y}^*$ of the following bounded least square problem

$$\begin{aligned} &\min \|A\mathbf{y} - \mathbf{b}\|^2 \\ &\text{subject to } \mathbf{y} \geq \mathbf{0}\,, \end{aligned} \tag{2}$$

(see Appendix A). As we shall see, the fact that this algorithm establishes the existence of a point $\mathbf{y}^*$ that solves (2), is the key to compute either a state price vector, if non-arbitrage opportunities exist, or an arbitrage opportunity, if there are no state price vectors. On the other hand and for a practical purpose, the above argument justify the use, for example, of the `lsqnonneg` function of the MATLAB Optimization Toolbox (this function gives a numerical solution of the BLS problem (2)).

Now we can give some preliminary definitions and results about basic Linear Algebra. Let A be an $m \times n$ matrix. Then we define the column space of A, which we denote by col A, as

$$\text{col } A = \text{span}\{A\mathbf{e}_1, A\mathbf{e}_2, \ldots, A\mathbf{e}_n\}\,,$$

where $\mathbf{e}_i$ denotes the ith column of the $n \times n$ identity matrix. In particular, if we set

$$\mathbf{S}^i = \mathbf{S}_{\Delta t}\mathbf{e}_i$$

for $i = 1, 2, \ldots, k$, then

$$\text{col } \mathbf{S}_{\Delta t} = \text{span }\{\mathbf{S}^1, \mathbf{S}^2, \ldots, \mathbf{S}^k\}\,.$$

In a similar way as above we define the row space of A, denoted by row A, by

$$\text{row } A = \text{col } A'.$$

Let

$$\text{nul } A = \{\mathbf{x} \,:\, A\mathbf{x} = \mathbf{0}\}\,,$$

and for a vectorial subspace $E \subset \mathbb{R}^n$, we shall denote by $E^{\perp}$ the orthogonal complement of E, that is,

$$E^{\perp} = \{\mathbf{x} \,:\, \mathbf{x}'\mathbf{y} = \mathbf{0} \text{ for all } \mathbf{y} \in E\}\,.$$

It is well known (Strang, 1998) that

$$E \cap E^{\perp} = \{\mathbf{0}\}\,,$$

and for all $\mathbf{x} \in \mathbb{R}^n$ there exist $\mathbf{x}_1 \in E$ and $\mathbf{x}_2 \in E^{\perp}$ such that

$$\mathbf{x} = \mathbf{x}_1 + \mathbf{x}_2\,.$$

Moreover,

$$(\text{nul } A)^{\perp} = \text{row } A = \text{col } A'.$$

Finally, set $\mathbb{K} = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} \geq \mathbf{0}\}$, $\mathring{\mathbb{K}} = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} > \mathbf{0}\}$ and for $\mathbf{x} \in \mathbb{R}^n$, let $\mathcal{Z}(\mathbf{x}) = \{i : x_i = 0\}$.

THEOREM 2. *Let Ψ^* be a solution of*

$$\begin{aligned} &\min \|\mathbf{S}_{\Delta t}\Psi - \mathbf{S}_0\|^2 \\ &\text{subject to } \Psi \geq \mathbf{0}\,, \end{aligned} \tag{3}$$

and take the residual vector $\boldsymbol{\theta}^* = \mathbf{S}_{\Delta t}\Psi^* - \mathbf{S}_0$. If $\boldsymbol{\theta}^* \neq \mathbf{0}$ then $\boldsymbol{\theta}^*$ satisfies (Arb 2). Otherwise, if $\boldsymbol{\theta}^* = \mathbf{0}$ then one and only one of the following statements holds:

1. *If $\Psi^* > \mathbf{0}$ then there are no arbitrage opportunities.*
2. *If $\Psi^* \geq \mathbf{0}$ and*

   $$\text{span}\,\{\mathbf{S}^i : i \in \mathcal{Z}(\Psi^*)\} \subset \text{span}\,\{\mathbf{S}^i : i \notin \mathcal{Z}(\Psi^*)\}\,,$$

   then there exist $\delta > 0$ and a continuous path of state price vectors Ψ^*_ε, where $\varepsilon \in (0, \delta)$, and such that

   $$\lim_{\varepsilon \to 0} \Psi^*_\varepsilon = \Psi^*.$$

   Moreover, there are no arbitrage opportunities.
3. *If $\Psi^* \geq \mathbf{0}$ and*

   $$\text{span}\,\{\mathbf{S}^i : i \in \mathcal{Z}(\Psi^*)\} \not\subseteq \text{span}\,\{\mathbf{S}^i : i \notin \mathcal{Z}(\Psi^*)\}\,,$$

   then there are arbitrage opportunities which satisfy (Arb 1) and there are no state price vectors. Moreover, let $\mathbf{y}^*$ be a solution of

   $$\begin{aligned} &\min \|[\mathbf{S}_{\Delta t}, -\mathbf{S}_0]\mathbf{y} + [\mathbf{S}_{\Delta t}, -\mathbf{S}_0]\mathbf{e}\|^2 \\ &\text{subject to } \mathbf{y} \geq \mathbf{0}\,, \end{aligned} \tag{4}$$

   where $\mathbf{e} = [1, 1, \ldots, 1]'$, then

   $$\boldsymbol{\theta}^* = [\mathbf{S}_{\Delta t}, -\mathbf{S}_0]\mathbf{y}^* + [\mathbf{S}_{\Delta t}, -\mathbf{S}_0]\mathbf{e}$$

   is an arbitrage opportunity.

From the above theorem we obtain the following result.

COROLLARY 1. *Theorem 2 implies Theorem 1.*

From Theorem 2 we obtain that to construct either a state price vector or an arbitrage opportunity, we only need to show that a BLS of type (2) always has a solution. To justify this argument and for completeness, in Appendix A we give an algorithm due to Dax (1997).

To end this section, four numerical examples are given, one for each statement, to illustrate the usefulness of this constructive approach. Hence, the `lsqnonneg` function, of the MATLAB Optimization Toolbox, will be used in order to solve either (3) or (4).

In the following three examples we consider $\mathbf{S}_{\Delta t}$ equal to

$$\begin{bmatrix} 1.00 & 1.00 & 1.00 & 1.00 & 1.00 & 1.00 & 1.00 & 1.00 & 1.00 & 1.00 \\ 1.05 & 1.38 & 2.23 & 4.20 & 2.56 & 3.15 & 1.35 & 2.00 & 3.02 & 0.99 \\ 1.20 & 1.24 & 2.56 & 4.23 & 2.56 & 3.65 & 1.25 & 2.58 & 3.50 & 0.96 \\ 1.23 & 1.45 & 2.34 & 4.26 & 2.56 & 3.25 & 1.99 & 1.99 & 3.14 & 0.89 \\ 1.34 & 1.23 & 2.67 & 4.89 & 2.09 & 3.87 & 1.95 & 2.06 & 3.65 & 1.02 \\ 1.65 & 1.24 & 2.89 & 4.09 & 2.89 & 2.58 & 1.80 & 2.42 & 3.51 & 1.50 \\ 1.45 & 1.65 & 2.90 & 4.23 & 2.09 & 3.00 & 2.50 & 2.36 & 3.40 & 0.95 \\ 1.38 & 1.98 & 2.98 & 4.56 & 2.78 & 3.65 & 1.64 & 2.36 & 2.99 & 0.97 \\ 1.86 & 1.27 & 2.99 & 4.89 & 2.31 & 3.25 & 1.23 & 2.00 & 2.57 & 1.23 \\ 1.56 & 1.21 & 2.67 & 4.12 & 2.24 & 3.14 & 1.00 & 1.55 & 3.05 & 1.00 \end{bmatrix}$$

In the first one, let

$$\mathbf{S}_0 = \begin{bmatrix} 3.03 \\ 2.08 \\ 3.09 \\ 2.21 \\ 3.01 \\ 2.87 \\ 1.32 \\ 0.98 \\ 4.01 \\ 3.64 \end{bmatrix} \text{ and we obtain } \Psi^* = \begin{bmatrix} 0.7086 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1.5664 \end{bmatrix} \text{ and } \boldsymbol{\theta}^* = \begin{bmatrix} 0.7550 \\ -0.2148 \\ 0.7359 \\ -0.0557 \\ 0.4627 \\ -0.6488 \\ -1.1956 \\ -1.5173 \\ 0.7653 \\ 0.9682 \end{bmatrix}.$$

Therefore, from the main statement of Theorem 2, the residual vector $\boldsymbol{\theta}^*$ provides an arbitrage opportunity.

Now, let

$$\mathbf{S}_0 = \begin{bmatrix} 4.0500 \\ 8.4056 \\ 9.1860 \\ 8.8618 \\ 9.0691 \\ 9.3665 \\ 9.6371 \\ 9.9531 \\ 8.4859 \\ 7.6952 \end{bmatrix}. \text{ Then, we obtain } \Psi^* = \begin{bmatrix} 0.10 \\ 0.90 \\ 0.30 \\ 0.20 \\ 0.45 \\ 0.23 \\ 0.45 \\ 0.96 \\ 0.34 \\ 0.12 \end{bmatrix} \text{ and } \|\boldsymbol{\theta}^*\| \approx 0\,.$$

From the statement 1 of Theorem 2, we have that Ψ^* is a state price vector.

To illustrate the statement 3 of Theorem 2, let

$$\mathbf{S}_0 = \begin{bmatrix} 2.7700 \\ 6.0177 \\ 6.1615 \\ 6.5499 \\ 6.6203 \\ 6.8336 \\ 6.7538 \\ 7.0184 \\ 6.6595 \\ 5.9046 \end{bmatrix}. \text{ Then, we obtain } \Psi^* = \begin{bmatrix} 0.54 \\ 0.21 \\ 0.23 \\ 0.41 \\ 0.70 \\ 0.12 \\ 0.56 \\ 0 \\ 0 \\ 0 \end{bmatrix} \text{ and } \|\boldsymbol{\theta}^*\| \approx 0\,.$$

Then we compute the reduced row echelon form of $\mathbf{S}_{\Delta t}$ (using the MATLAB function `rref`) obtaining that it is equal to the identity matrix. In consequence we have that the set of the first seven columns of $\mathbf{S}_{\Delta t}$ is linearly independent of the set given by the last three ones. By using the statement 3 of Theorem 2, we obtain the existence of arbitrage opportunities. Now we compute them by solving (4), and we obtain as residual vector

$$\boldsymbol{\theta}^* = \begin{bmatrix} -0.1661 \\ -0.4621 \\ -0.1553 \\ 0.5028 \\ -0.0628 \\ -0.0189 \\ -0.1405 \\ 0.1949 \\ -0.0290 \\ 0.2072 \end{bmatrix},$$

which gives an arbitrage opportunity.

Finally, we construct the following example of statement 2 of Theorem 2. Let be

$$\mathbf{S}_{\Delta t} = \begin{bmatrix} 1.0 & 1.0 & 1.0 & 1.0 & 1.0 & 1.0 \\ 2.0 & 2.1 & 1.9 & 1.8 & 2.5 & 2.4 \\ 3.0 & 2.8 & 2.9 & 3.3 & 3.2 & 2.7 \\ 1.0 & 1.2 & 1.9 & 2.2 & 2.5 & 1.7 \end{bmatrix} \text{ and } \mathbf{S}_0 = \begin{bmatrix} 1 \\ 2.2 \\ 3.1 \\ 1.8 \end{bmatrix}.$$

By solving (3) we obtain

$$\Psi^* = \begin{bmatrix} 0.4075 \\ 0 \\ 0 \\ 0.1285 \\ 0.4013 \\ 0.0627 \end{bmatrix}.$$

Since

$$\mathbf{S}^2, \mathbf{S}^3 \in \text{span}\,\{\mathbf{S}^1, \mathbf{S}^4, \mathbf{S}^5, \mathbf{S}^6\}$$

it follows, by solving numerically

$$[\mathbf{S}^1\mathbf{S}^4\mathbf{S}^5\mathbf{S}^6]\mathbf{x} = \mathbf{S}^i \quad \text{for} \quad i = 2, 3,$$

that

$$\mathbf{S}^2 \approx 0.51411\mathbf{S}^1 + 0.11599\mathbf{S}^4 - 0.24765\mathbf{S}^5 + 0.61755\mathbf{S}^6$$

and

$$\mathbf{S}^3 \approx -0.25392\mathbf{S}^1 + 0.91223\mathbf{S}^4 - 0.54232\mathbf{S}^5 + 0.88401\mathbf{S}^6.$$

Then, we take

$$\Psi_\varepsilon = \begin{bmatrix} 0.4075 - \varepsilon(0.51411 - 0.25392) \\ \varepsilon \\ \varepsilon \\ 0.1285 - \varepsilon(0.11599 + 0.91223) \\ 0.4013 + \varepsilon(0.24765 + 0.54232) \\ 0.0627 - \varepsilon(0.61755 + 0.88401) \end{bmatrix}.$$

If $\varepsilon \in (0, 4.1757\times10^{-2})$ then, it is not difficult to see that, $\Psi_\varepsilon > \mathbf{0}$ and $\mathbf{S}_{\Delta t}\Psi_\varepsilon \approx \mathbf{S}_0$.

3. Proof of the Main Result

This section is devoted to the proof of Theorem 2. To this end we will reduce the problem to compute the solution of a BLS problem. More precisely, we will start considering the solution, namely $\Psi^* \in \mathbb{R}^k$, of (3). Then, if the residual vector of $\boldsymbol{\theta}^*$ is different from $\mathbf{0}$ it gives an arbitrage opportunity. Otherwise, we have only two possibilities depending on Ψ^*. If $\Psi^* > \mathbf{0}$, then Ψ^* is a state price vector. Otherwise, if $\Psi^* \geq \mathbf{0}$, then we will take a partition in the set of all future price vectors. This partition will be given by the set of future prices vectors associated with the entries of Ψ^* equal to 0, and its complementary. Also, in this situation, we have only two possibilities given by the fact that the first set can be either linearly dependent or linearly independent of the second one. If linear dependence holds, then we will construct a continuous path of state price vectors. Otherwise, there exists an arbitrage opportunity and we will obtain it by computing the residual vector of (4).

Assume that $\Psi^* \geq \mathbf{0}$ is a solution of (3) obtained by using the Dax's Algorithm given in the Appendix A. If $\boldsymbol{\theta}^* \neq \mathbf{0}$, then we will use the following useful lemma due to Dax (1993), Lemma 2 (see Appendix A).

LEMMA 1. *Let $\Psi^* \in \mathbb{R}^k$. Then Ψ^* holds (3) if and only if Ψ^* and $\boldsymbol{\theta}^*$ satisfy*

$$\Psi^* \geq \mathbf{0}, \mathbf{S}_{\Delta t}\boldsymbol{\theta}^* \geq \mathbf{0} \quad \textit{and} \quad (\Psi^*)'\mathbf{S}'_{\Delta t}\boldsymbol{\theta}^* = 0\,. \tag{5}$$

By using the above lemma, we have that

$$\begin{aligned} \mathbf{S}'_0\boldsymbol{\theta}^* &= (\mathbf{S}_{\Delta t}\Psi^* - \boldsymbol{\theta}^*)^T\boldsymbol{\theta}^* \\ &= (\Psi^*)'\mathbf{S}'_{\Delta t}\boldsymbol{\theta}^* - (\boldsymbol{\theta}^*)'\boldsymbol{\theta}^* \\ &= -\|\boldsymbol{\theta}^*\|^2 < 0\,. \end{aligned}$$

Since, $\mathbf{S}'_{\Delta t}\boldsymbol{\theta}^* \geq \mathbf{0}$, we obtain that $\boldsymbol{\theta}^*$ is an arbitrage opportunity which satisfies (Arb 2).

Now, assume that $\boldsymbol{\theta}^* = \mathbf{0}$. Then we will consider the following two situations. First, assume that $\Psi^* > \mathbf{0}$. In this case Ψ^* is a state price vector. We claim that there are no arbitrage opportunities. Otherwise, there exists $\boldsymbol{\theta} \in \mathbb{R}^N$ satisfying that

$$\mathbf{S}'_0\boldsymbol{\theta} = (\mathbf{S}_{\Delta t}\Psi^*)'\boldsymbol{\theta} = (\Psi^*)'\mathbf{S}'_{\Delta t}\boldsymbol{\theta} > 0\,,$$

a contradiction and the claim follows. Thus statement 1 holds.

Now, assume that $\Psi^* \geq \mathbf{0}$. In this case we can write, without loss of generality, that

$$\Psi^* = [0, \ldots, 0, \psi^*_{s+1}, \ldots, \psi^*_k]'$$

for some $s > 0$. We recall that $\mathbf{S}^i = \mathbf{S}_{\Delta t}\mathbf{e}_i$. From now on we will denote by $\mathbf{S}_{\Delta t,2}$ the matrix given by

$$[\mathbf{S}^{s+1}\mathbf{S}^{s+2}\cdots\mathbf{S}^k]\,.$$

Now, assume that there exists $1 \leq l \leq k - s$ be such that

$$\text{col}\ \mathbf{S}_{\Delta t,2} = \text{span}\ \{\mathbf{S}^{k-l+1}, \mathbf{S}^{k-l+2}, \ldots, \mathbf{S}^k\}$$

and where $\{\mathbf{S}^{k-l+1}, \mathbf{S}^{k-l+2}, \ldots, \mathbf{S}^{k}\}$ is a set of linearly independent vectors. Then

$$\mathbf{S}^{i} = \sum_{j=k-l+1}^{k} \lambda_{i,j} \mathbf{S}^{j}, \tag{6}$$

where $\lambda_{ij} \in \mathbb{R}$, for $i = 1, 2, \ldots, k-l$ and $j = k-l+1, \ldots, k$. Moreover,

$$\mathbf{S}_0 = \mathbf{S}'_{\Delta t} \Psi^* = \sum_{j=s+1}^{k} \psi_j^* \mathbf{S}^{j}. \tag{7}$$

Finally, we take $\mathbf{S}_{\Delta t,1} = [\mathbf{S}^1 \mathbf{S}^2 \cdots \mathbf{S}^s]$. Note that $\mathbf{S}_{\Delta t} = [\mathbf{S}_{\Delta t,1} \mathbf{S}_{\Delta t,2}]$. The next lemma will be useful to prove statement 2.

LEMMA 2. *If* col $\mathbf{S}_{\Delta t,1} \subset$ col $\mathbf{S}_{\Delta t,2}$, *then there exist* $\delta > 0$ *and a continuous path of state price vectors* Ψ_ε^*, *where* $\varepsilon \in (0, \delta)$, *and such that*

$$\lim_{\varepsilon \to 0} \Psi_\varepsilon^* = \Psi^*.$$

Proof. To prove the lemma we need to find $\Psi = [\psi_1, \ldots, \psi_k]' > \mathbf{0}$ such that

$$\mathbf{S}_0 = \sum_{j=s+1}^{k} \psi_j^* \mathbf{S}^{j} = \sum_{i=1}^{k} \psi_i \mathbf{S}^{i}. \tag{8}$$

Then, we can write

$$\begin{aligned}
\sum_{i=1}^{k} \psi_i \mathbf{S}^{i} &= \sum_{i=1}^{s} \psi_i \mathbf{S}^{i} + \sum_{t=s+1}^{k-l} \psi_t \mathbf{S}^{t} + \sum_{j=k-l+1}^{k} \psi_j \mathbf{S}^{j} \\
&= \sum_{j=k-l+1}^{k} \left(\sum_{i=1}^{s} \psi_i \lambda_{i,j} \right) \mathbf{S}^{j} + \sum_{t=s+1}^{k-l} \psi_t \mathbf{S}^{t} + \sum_{j=k-l+1}^{k} \psi_j \mathbf{S}^{j} \\
&= \sum_{t=s+1}^{k-l} \psi_t \mathbf{S}^{t} + \sum_{j=k-l+1}^{k} \left(\sum_{i=1}^{s} \psi_i \lambda_{i,j} - \psi_j \right) \mathbf{S}^{j}.
\end{aligned}$$

By using (8) we obtain that,

$$\sum_{j=s+1}^{k} \psi_j^* \mathbf{S}^{j} = \sum_{t=s+1}^{k-l} \psi_t \mathbf{S}^{t} + \sum_{j=k-l+1}^{k} \left(\sum_{i=1}^{s} \psi_i \lambda_{i,j} - \psi_j \right) \mathbf{S}^{j}.$$

Finally, Ψ must hold that $\psi_t = \psi_t^*$ for $t = s+1, \ldots, k-l$ and

$$\left(\sum_{i=1}^{s} \psi_i \lambda_{i,j} \right) + \psi_j = \psi_j^*$$

for $j = k - l + 1, \ldots, s$. If we take $\psi_i = \varepsilon > 0$ for $i = 1, 2, \ldots, s$, then

$$\varepsilon \left(\sum_{i=1}^{s} \lambda_{i,j} \right) + \psi_j = \psi_j^*$$

for $j = k - l + 1, \ldots, s$. Thus, we only need to choose $\varepsilon > 0$ satisfying that

$$\psi_j = \psi_j^* - \varepsilon \left(\sum_{i=1}^{s} \lambda_{i,j} \right) > 0 \tag{9}$$

for all $j = k - l + 1, \ldots, s$. To see this we consider the set

$$\mathcal{K} = \left\{ j \in \{k - l + 1, \ldots, s\} : \sum_{i=1}^{s} \lambda_{i,j} > 0 \right\} ,$$

and we take

$$\delta = \min \left\{ \frac{\psi_j^*}{\sum_{i=1}^{s} \lambda_{i,j}} : j \in \mathcal{K} \right\} > 0 .$$

Therefore, for all $\varepsilon < \delta$, the equality (9) holds for $j = k - l + 1, \ldots, s$. Finally, we conclude the proof of lemma considering Ψ_ε^* as

$$\left[\varepsilon, \ldots, \varepsilon, \psi_{s+1}^*, \ldots, \psi_{k-l}^*, \psi_{k-l+1}^* - \varepsilon \left(\sum_{i=1}^{s} \lambda_{i,k-l+1} \right), \ldots, \psi_k^* - \varepsilon \left(\sum_{i=1}^{s} \lambda_{i,k} \right) \right]' .$$

Clearly Ψ_ε^* is a state price vector for all $\varepsilon \in (0, \delta)$ satisfying that $\lim_{\varepsilon \to 0} \Psi_\varepsilon^* = \Psi^*$. ■

We remark that, from the above lemma and the proof of statement 1, if $\Psi^* \geq \mathbf{0}$ and

$$\operatorname{span} \{\mathbf{S}^i : i \in \mathcal{Z}(\Psi^*)\} = \operatorname{col} \mathbf{S}_{\Delta t} ,$$

then there are no arbitrage opportunities. Thus, statement 2 follows.

Finally, in order to prove statement 3, suppose that

$$\operatorname{col} \mathbf{S}_{\Delta t,1} \nsubseteq \operatorname{col} \mathbf{S}_{\Delta t,2} .$$

Then $s > 1$ and, without loss of generality, we may assume the existence of a maximal integer number $l^* \geq 0$, with $s - l^* > 1$, such that

$$\mathbf{S}^{s-l^*}, \mathbf{S}^{s-l^*+1}, \ldots, \mathbf{S}^s \in \operatorname{col} \mathbf{S}_{\Delta t,1} \cap \operatorname{col} \mathbf{S}_{\Delta t,2} .$$

Now we take

$$\mathbf{S}_{\Delta t,1}^* = [\mathbf{S}^1 \ldots \mathbf{S}^{s-l^*-1}]$$

and

$$\mathbf{S}^*_{\Delta t,2} = [\mathbf{S}^{s-l^*} \dots \mathbf{S}^s \mathbf{S}^{s+1} \dots \mathbf{S}^k] \, .$$

Note that $\mathbf{S}_{\Delta t} = [\mathbf{S}^*_{\Delta t,1}\mathbf{S}^*_{\Delta t,2}]$ and $\mathrm{col}\, \mathbf{S}^*_{\Delta t,1} \cap \mathrm{col}\, \mathbf{S}^*_{\Delta t,2} = \{\mathbf{0}\}$. Moreover, from the same argument used to prove Lemma 2, we can construct a vector $\Psi^*_2 > \mathbf{0}$ satisfying that

$$\mathbf{S}^*_{\Delta t,2}\Psi^*_2 = \mathbf{S}_0 \, ,$$

that is,

$$[\mathbf{S}^*_{\Delta t,1}\mathbf{S}^*_{\Delta t,2}] \begin{bmatrix} \mathbf{0} \\ \Psi^*_2 \end{bmatrix} = \mathbf{S}_0 \, .$$

In this context, the following lemma gives a characterization of the existence of arbitrage opportunities.

LEMMA 3. *Let*

$$E = \{\mathbf{X} : \mathbf{X} = (\mathbf{S}^*_{\Delta t,1})'\boldsymbol{\theta} \textit{ for some } \boldsymbol{\theta} \in \mathrm{nul}\, (\mathbf{S}^*_{\Delta t,2})'\} \, .$$

Then $E \cap \mathbb{K} \neq \{\mathbf{0}\}$ *if and only if there are arbitrage opportunities satisfying (Arb 1)*

Proof. Note that if $\mathbf{X} = (\mathbf{S}^*_{\Delta t,1})'\boldsymbol{\theta} \in E \cap \mathbb{K}$ and $E \cap \mathbb{K} \neq \{\mathbf{0}\}$, then

$$\mathbf{S}'_{\Delta t}\boldsymbol{\theta} = \begin{bmatrix} (\mathbf{S}^*_{\Delta t,1})' \\ (\mathbf{S}^*_{\Delta t,2})' \end{bmatrix} \boldsymbol{\theta} \geq \mathbf{0} \, .$$

Since

$$(\mathrm{nul}\, (\mathbf{S}^*_{\Delta t,2})')^{\perp} = \mathrm{col}\, \mathbf{S}^*_{\Delta t,2}$$

and $\mathbf{S}_0 \in \mathrm{col}\, \mathbf{S}^*_{\Delta t,2}$, then

$$\mathbf{S}'_0\boldsymbol{\theta} = \mathbf{0} \, .$$

Thus, $\boldsymbol{\theta}$ is an arbitrage opportunity satisfying (Arb 1). Conversely, if $\boldsymbol{\theta}$ is an arbitrage opportunity satisfying (Arb 1) then

$$\mathbf{S}'_0\boldsymbol{\theta} = [\mathbf{0}(\Psi^*_2)'] \begin{bmatrix} (\mathbf{S}^*_{\Delta t,1})' \\ (\mathbf{S}^*_{\Delta t,2})' \end{bmatrix} \boldsymbol{\theta} = \mathbf{0}$$

and

$$\begin{bmatrix} (\mathbf{S}^*_{\Delta t,1})' \\ (\mathbf{S}^*_{\Delta t,2})' \end{bmatrix} \boldsymbol{\theta} \geq \mathbf{0} \, .$$

Therefore,

$$(\mathbf{S}^*_{\Delta t,1})'\boldsymbol{\theta} \geq \mathbf{0} \quad \text{and} \quad (\mathbf{S}^*_{\Delta t,2})'\boldsymbol{\theta} = \mathbf{0} \, ,$$

and, in consequence, $(\mathbf{S}^*_{\Delta t,1})'\boldsymbol{\theta} \in E \cap \mathbb{K}$. This ends the proof of lemma. ■

LEMMA 4. $E^{\perp} = \text{nul}\, \mathbf{S}^*_{\Delta t,1}$.

Proof. Let $\mathbf{Y} \in E^{\perp}$. Then $\mathbf{Y}'\mathbf{X} = 0$ for all $\mathbf{X} \in E$. Thus, $\mathbf{Y}'(\mathbf{S}^*_{\Delta t,1})'\boldsymbol{\theta} = 0$ for all $\boldsymbol{\theta} \in \text{nul}\,(\mathbf{S}^*_{\Delta t,2})'$, that is, $(\mathbf{S}^*_{\Delta t,1}\mathbf{Y})'\boldsymbol{\theta} = 0$ for all $\boldsymbol{\theta} \in \text{nul}\,(\mathbf{S}^*_{\Delta t,2})'$. Therefore, $\mathbf{S}^*_{\Delta t,1}\mathbf{Y} \in (\text{nul}\,(\mathbf{S}^*_{\Delta t,2})') = \text{col}\,\mathbf{S}^*_{\Delta t,2}$. Since $\text{col}\,\mathbf{S}^*_{\Delta t,1} \cap \text{col}\,\mathbf{S}^*_{\Delta t,2} = \{\mathbf{0}\}$, we have that $\mathbf{S}^*_{\Delta t,1}\mathbf{Y} = \mathbf{0}$, and $\mathbf{Y} \in \text{nul}\,\mathbf{S}^*_{\Delta t,1}$. On the other hand, if $\mathbf{S}^*_{\Delta t,1}\mathbf{Y} = \mathbf{0}$ then $\mathbf{Y}'(\mathbf{S}^*_{\Delta t,1})'\boldsymbol{\theta} = 0$ for all $\boldsymbol{\theta} \in \text{nul}\,(\mathbf{S}^*_{\Delta t,2})'$. Therefore, $\mathbf{Y} \in E^{\perp}$ and the lemma follows. ■

Under the assumptions of statement 3, the following lemma gives a characterization of the existence of a state price vector.

LEMMA 5. $\text{nul}\,\mathbf{S}^*_{\Delta t,1} \cap \overset{\circ}{\mathbb{K}} \neq \emptyset$ *if and only if there exists a state price vector.*

Proof. If $\text{nul}\,\mathbf{S}^*_{\Delta t,1} \cap \overset{\circ}{\mathbb{K}} \neq \emptyset$, then there exists $\Psi_1 > 0$ such that $\mathbf{S}^*_{\Delta t,1}\Psi_1 = 0$. Thus,

$$[\mathbf{S}^*_{\Delta t,1}\mathbf{S}^*_{\Delta t,2}]\begin{bmatrix}\Psi_1\\ \Psi_2^*\end{bmatrix} = \mathbf{S}_0\,,$$

that is, $\Psi = [\Psi_1\ \Psi_2^*]'$ is a state price vector. Conversely, let $\Psi = [\Psi_1\ \Psi_2]'$ be a state price vector. Then

$$[\mathbf{S}^*_{\Delta t,1}\mathbf{S}^*_{\Delta t,2}]\begin{bmatrix}\Psi_1\\ \Psi_2\end{bmatrix} = \mathbf{S}_0\,,$$

that is,

$$\mathbf{S}^*_{\Delta t,1}\Psi_1 = \mathbf{S}_0 - \mathbf{S}^*_{\Delta t,2}\Psi_2 \in \text{col}\,\mathbf{S}^*_{\Delta t,2}\,.$$

By the fact that $\text{col}\,\mathbf{S}^*_{\Delta t,1} \cap \text{col}\,\mathbf{S}^*_{\Delta t,2} = \{\mathbf{0}\}$, we have that

$$\mathbf{S}^*_{\Delta t,1}\Psi_1 = \mathbf{0}\,.$$

Therefore, $\Psi_1 \in \text{nul}\,\mathbf{S}^*_{\Delta t,1} \cap \overset{\circ}{\mathbb{K}}$ and the lemma follows. ■

From the separating hyperplane theorem and the Riesz's Lemma the following result follows.

LEMMA 6. *Let F be a subspace of $\mathbb{R}^n$ such that $F^{\perp} \neq \{\mathbf{0}\}$. If $F \cap \mathbb{K} = \{\mathbf{0}\}$ then there exists $\mathbf{y}^* \in F^{\perp}$ such that $\mathbf{x}'\mathbf{y}^* > 0$ for all $\mathbf{x} \in \mathbb{K}$ that is $F^{\perp} \cap \mathbb{K} \neq \{\mathbf{0}\}$.*

Next, we prove the first part of statement 3. To this end, assume that there is a state price vector. From Lemma 5 we have that $\text{nul}\,\mathbf{S}^*_{\Delta t,1} \cap \overset{\circ}{\mathbb{K}} \neq \emptyset$. Thus, there exists $\Psi_1 > \mathbf{0}$ satisfying $\mathbf{S}^*_{\Delta t,1}\Psi_1 = \mathbf{0}$, a contradiction because $\mathbf{S}^*_{\Delta t,1} \geq \mathbf{0}$. Now, since $\operatorname{nul} \mathbf{S}^*_{\Delta t,1} \cap \overset{\circ}{\mathbb{K}} = \emptyset$, that is $E^\perp \cap \overset{\circ}{\mathbb{K}} = \emptyset$, then either $E^\perp \cap \partial\mathbb{K} \neq \emptyset$ or $E^\perp \cap \partial\mathbb{K} = \emptyset$ holds, where $\partial\mathbb{K}$ denotes the topological boundary of $\mathbb{K}$. It is not difficult to see that if $E^\perp \cap \partial\mathbb{K} \neq \emptyset$ then $E \cap \mathbb{K} \neq \{\mathbf{0}\}$ and, from Lemma 3, we have the existence of arbitrage opportunities satisfying (Arb 1). On the other hand, assume that $E^\perp \cap \partial\mathbb{K} = \emptyset$, that is $E^\perp \cap \mathbb{K} = \{\mathbf{0}\}$, then, by using Lemma 6, $E \cap \mathbb{K} \neq \{\mathbf{0}\}$ and in a similar way as above we obtain the existence of arbitrage opportunities. This ends the proof of the first part of statement 3.

To prove the second part we shall use the following two lemmas. The proof of the first one is straightforward.

LEMMA 7. *There exists a state price vector if and only if there exists* $\mathbf{y}^* > \mathbf{0}$ *such that*

$$[\mathbf{S}_{\Delta t}, -\mathbf{S}_0]\mathbf{y}^* = \mathbf{0}\,.$$

By using Stiemkes's Theorem (Dax, 1993) we have the following result.

LEMMA 8. *Either the system*

$$[\mathbf{S}_{\Delta t}, -\mathbf{S}_0]\mathbf{y}^* = \mathbf{0} \quad \textit{and} \quad \mathbf{y}^* > \mathbf{0} \tag{10}$$

has a solution $\mathbf{y}^*$*, or the system*

$$[\mathbf{S}_{\Delta t}, -\mathbf{S}_0]'\boldsymbol{\theta} \geq \mathbf{0}, \quad [\mathbf{S}_{\Delta t}, -\mathbf{S}_0]'\boldsymbol{\theta} \neq \mathbf{0} \tag{11}$$

has a solution $\boldsymbol{\theta}$*, but never both.*

An equivalent way to write (11) is

$$[\mathbf{S}_{\Delta t}, -\mathbf{S}_0]'\boldsymbol{\theta} \geq \mathbf{0} \quad \text{and} \quad \mathbf{e}'[\mathbf{S}_{\Delta t}, -\mathbf{S}_0]'\boldsymbol{\theta} < 0\,, \tag{12}$$

where $\mathbf{e} = [1, 1, \ldots, 1]'$. Moreover, if $\boldsymbol{\theta}$ satisfies (11), then $\boldsymbol{\theta}$ is an arbitrage opportunity. Since there are no state price vectors, from Lemmas 7 and 8, there exists $\boldsymbol{\theta}$ satisfying (12). From Dax (1993) Theorem 1.1, it follows that if $\mathbf{y}^*$ is a solution of

$$\begin{aligned} &\min \|[\mathbf{S}_{\Delta t}, -\mathbf{S}_0]\mathbf{y} + [\mathbf{S}_{\Delta t}, -\mathbf{S}_0]\mathbf{e}\|^2 \\ &\text{subject to } \mathbf{y} \geq \mathbf{0}\,, \end{aligned} \tag{13}$$

and

$$\boldsymbol{\theta}^* = [\mathbf{S}_{\Delta t}, -\mathbf{S}_0]\mathbf{y}^* + [\mathbf{S}_{\Delta t}, -\mathbf{S}_0]\mathbf{e}$$

is different from zero, then $\boldsymbol{\theta}^*$ solves (12). Note that if $\boldsymbol{\theta}^* = \mathbf{0}$ then

$$[\mathbf{S}_{\Delta t}, -\mathbf{S}_0](\mathbf{y}^* + \mathbf{e}) = \mathbf{0}\,,$$

and, from Lemma 7, $\mathbf{y}^* + \mathbf{e}$ is a state price vector, a contradiction. Thus, $\boldsymbol{\theta}^* \neq \mathbf{0}$ is an arbitrage opportunity. Finally, by using Dax's Algorithm given in Appendix A, we compute $\mathbf{y}^*$ which solves (13).

Appendix A. Dax's Algorithm and the Proof of Lemma 1

First of all we introduce a simple iterative algorithm due to Dax (1993) in order to establish the existence of a point $\mathbf{y}^*$ that solves (2). Moreover, it is possible to show (Dax, 1997) that the algorithm ends in a finite number of iterations.

THE DAX ALGORITHM

Assume that

$$A = [\mathbf{A}^1 \mathbf{A}^2 \ldots \mathbf{A}^k],$$

where $\mathbf{A}^i = A\mathbf{e}_i$. We proceed by its ith iteration, $i = 1, 2, \ldots,$ which consists of the following two steps.

Step 1: Let $\mathbf{y}_i = [y_1, y_2, \ldots, y_k]' \geq \mathbf{0}$ denote the current estimate of the solution beginning of the ith iteration. Define

$$\mathbf{r}_i = A\mathbf{y}_i - \mathbf{b}.$$

If Cardinal $\mathcal{Z}(\mathbf{y}_i)^c = 0$, where $\mathcal{Z}(\mathbf{y}_i)^c = \{j : y_j > 0\}$, or $\mathbf{r}_i = \mathbf{0}$ then skip to Step 2. Otherwise, let A_i, the matrix whose columns are $\mathbf{A}^l$, with $l \in \mathcal{Z}(\mathbf{y}_i)^c$. For simplicity we assume that

$$A_i = [\mathbf{A}^{s+1} \ldots \mathbf{A}^k], \mathcal{Z}(\mathbf{y}_i) = \{1, 2, \ldots, s\} \quad \text{and} \quad \mathcal{Z}(\mathbf{y}_i)^c = \{s+1, \ldots, k\}.$$

Let the vector $\mathbf{w}_i = [w_{s+1}, w_{s+2}, \ldots, w_k]'$ solve the unconstrained least squares problem

$$\min \|A_i \mathbf{w}_i - \mathbf{r}_i\|^2.$$

We note $\mathbf{0}$ solves this problem if and only if $A_i'\mathbf{r}_i = \mathbf{0}$. In this case skip to Step 2. Otherwise, define a nonzero search direction $\mathbf{u}_i = [u_1, u_2, \ldots, u_k]'$ by the following rule

$$u_l = 0 \text{ for } l = 1, \ldots, s \quad \text{and} \quad u_l = w_l \text{ for } l = s + \mathbf{1}, \ldots, k.$$

The next point is defined as

$$\mathbf{y}_{i+1} = \mathbf{y}_i + \nu_i \mathbf{u}_i,$$

where $\nu_i > 0$ is the largest number in the interval [0, 1] that keeps the point $\mathbf{y}_i + \nu_i \mathbf{u}_i$ feasible. In other words, ν_i is the smallest number in the set

$$\{1\} \cup \left\{ -\left(\frac{y_l}{u_l}\right) : u_l < 0 \quad \text{and} \quad l \in \{s+1, \ldots, k\} \right\}.$$

Step 2: In this step we have $A_i'\mathbf{r}_i = \mathbf{0}$ which means that $\mathbf{y}_i$ solves the problem

$$\min \|A\mathbf{y} - \mathbf{b}\|^2 \tag{A.1}$$

$$\text{subject to } y_l = 0 \text{ for } l \in \mathcal{Z}(\mathbf{y}) \tag{A.2}$$

and

$$y_l \geq 0 \text{ for } l \in \mathcal{Z}(\mathbf{y})^c. \tag{A.3}$$

To test whether or not $\mathbf{y}_i$ is optimal, we compute an index j such that

$$(\mathbf{A}^j)'\mathbf{r}_i = \min\{(\mathbf{A}^l)'\mathbf{r}_i : l \in \mathcal{Z}(\mathbf{y})\}\,.$$

If $(\mathbf{A}^j)'\mathbf{r}_i \geq 0$ then $\mathbf{y}_i$ and $\mathbf{r}_i$ satisfy (5). From Lemma 1 we have that $\mathbf{y}_i$ solves (2) and the algorithm ends in this case. Otherwise, the next point is defined as

$$\mathbf{y}_{i+1} = \mathbf{y}_i - \left(\frac{(\mathbf{A}^j)'\mathbf{r}_i}{(\mathbf{A}^j)'\mathbf{A}^j}\right)\mathbf{e}_j\,.$$

Note that

$$-\left(\frac{(\mathbf{A}^j)'\mathbf{r}_n}{(\mathbf{A}^j)'\mathbf{A}^j}\right) > 0\,,$$

and this point solves the problem

$$\min f(\lambda) = \|A(\mathbf{y}_n + \lambda\mathbf{e}_j) - \mathbf{b}\|^2. \qquad \blacksquare$$

Proof of Lemma 1. Assume that $\Psi^* = [\psi_1^*, \psi_2^*, \ldots, \psi_m^*]'$ solves (3) and let

$$f_i(\lambda) = \|\mathbf{S}_{\Delta t}(\Psi^* + \lambda\mathbf{e}_i) - \mathbf{S}_0\|^2 = \|\lambda\mathbf{S}^i - \boldsymbol{\theta}^*\|^2, \quad \text{for } i = 1, 2, \ldots, m\,.$$

Recall that $\mathbf{S}^i$ is the ith column vector of the matrix $\mathbf{S}_{\Delta t}$ and $\boldsymbol{\theta}^* = \mathbf{S}_{\Delta t}\Psi^* - \mathbf{S}_0$. Then, clearly, $\lambda = 0$ solves the problem

$$\begin{aligned}&\min f_i(\lambda)\\&\text{subject to } \psi_i^* + \lambda \geq 0\,.\end{aligned}$$

Therefore, since

$$\left.\frac{\mathrm{d}}{\mathrm{d}\lambda} f_i(\lambda)\right|_{\lambda=0} = 2(\mathbf{S}^i)'\boldsymbol{\theta}^*\,,$$

we have that $\psi_i^* > 0$ implies $2(\mathbf{S}^i)'\boldsymbol{\theta}^* = 0$, while $\psi_i^* = 0$ implies $2(\mathbf{S}^i)'\boldsymbol{\theta}^* \geq 0$, which constitutes

$$\Psi^* \geq \mathbf{0}, \mathbf{S}_{\Delta t}\boldsymbol{\theta}^* \geq \mathbf{0} \quad \text{and} \quad (\Psi^*)'\mathbf{S}_{\Delta t}'\boldsymbol{\theta}^* = 0\,. \tag{A.4}$$

Conversely, assume that the above conditions hold, and let Φ be an arbitrary point in $\mathbb{R}^m$ such that $\Phi \geq \mathbf{0}$. Let the vector $\Theta = [\eta_1, \eta_2, \ldots, \eta_m]'$ be obtained from Φ by the rule $\Theta = \Phi - \Psi^*$. Then $\psi_i^* = 0$ implies $\eta_i \geq 0$, while (A.4) leads to

$$\Theta' \mathbf{S}'_{\Delta t} \boldsymbol{\theta}^* \geq 0\,.$$

Hence, the identity

$$\|\mathbf{S}_{\Delta t}\Phi - \mathbf{b}\|^2 = \|\mathbf{S}_{\Delta t}\Psi^* - \mathbf{b}\|^2 + 2\Theta'\mathbf{S}'_{\Delta t}\boldsymbol{\theta}^* + \|\mathbf{S}_{\Delta t}\Theta^*\|^2$$

shows that

$$\|\mathbf{S}_{\Delta t}\Phi - \mathbf{b}\|^2 \geq \|\mathbf{S}_{\Delta t}\Psi^* - \mathbf{b}\|^2. \qquad \blacksquare$$

Acknowledgements

We thank the anonymous referee of the paper for careful reading and valuate suggestions which improved its presentation. This work has been partially supported by the DGES grant number PB96-1153 and by the Instituto Valenciano de Investigaciones Económicas.

References

Avellaneda, M. (1998). The minimum-entropy and related methods for calibrating asset-pricing models. *International Journal of Theoretical and Applied Finance*, submitted.

Bingham, N.H. and Kiesel, R. (1998). *Pricing and Hedging Financial Derivatives.* Springer Finance, Springer-Verlag.

Dax, A. (1993). The relationship between theorems of the alternative, least norm problems, steepest descent directions, and degeneracy: A review. *Ann. Oper. Res.*, **46**, 11–60.

Dax, A. (1997). An elementary proof of Farkas' Lemma. *SIAM Rev.*, **39**(3), 503–507.

Duffie, D. (1992). *Dynamic Asset Pricing Theory.* Princeton University Press.

Pliska, S.R. (1997). *Introduction to Mathematical Finance: Discrete Time Models.* Blackwell Publishers, Oxford.

Rockafellar, R. (1990). *Convex Analysis.* Princeton University Press.

Sheldon, Lin X. (1996). *Lecture Notes in Mathematical Finance.* Department of Statistics and Actuarial Science, University of Iowa. `http://www.stat.uiowa.edu/˜shlin/teaching.html`

Strang, G. (1998). *Introduction to Linear Algebra.* Wellesley-Cambridge Press.