Geometric Structures in Tensor Representations

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Abstract

In this paper we introduce a tensor subspace based format for the representation of a tensor in a topological tensor space. To do this we use a property of minimal subspaces which allow us to describe the tensor representation by means of a rooted tree. By using the tree structure and the dimensions of the associated minimal subspaces, we introduce the set of tensors in a tree based format with either bounded or fixed tree based rank. This class contains the Tucker format and the Hierarchical Tucker format (including the Tensor Train format). In particular, any tensor of the topological tensor space under consideration admits best approximations in the set of tensors in the tree based format with bounded tree based rank. Moreover, we show that the set of tensors in the tree based format with fixed tree based rank is an analytical Banach manifold. This local chart representation of the manifold is often crucial for an algorithmic treatment of high-dimensional time-dependent PDEs and minimisation problems. However, in our framework, the tangent (Banach) space at a given tensor is not a complemented subspace in the natural ambient tensor Banach space. Therefore, we study the differential of the natural inclusion map as a morphism between Banach manifolds. It allows us to discuss the Dirac-Frenkel variational principle in the framework of topological tensor spaces.

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1 Introduction

Tensor formats based on subspaces are widely used in scientific computation. Their constructions are usually based on a hierarchy of tensor product subspaces spanned by orthonormal bases, because in most cases one needs representations which are fitted to the special structure of the mathematical object under consideration.

Two of the most popular formats are the Tucker format and the Hierarchical Tucker format [11] (HT for short). It is possible to show that the Tensor Train format [21] (TT for short), introduced originally by Vidal [24], is a particular class of the HT format. An important feature of these formats, in the framework of topological tensor spaces, is the existence of a best approximation in each fixed set of tensors with bounded rank [6]. It allows to construct, on a theoretical level, iterative minimisation methods for nonlinear convex problems over reflexive tensor Banach spaces [7].

It is well-known that the Tucker format is also well applicable to the discretisation of differential equations in the framework of quantum chemical problems or of multireference Hartree and Hartree-Fock methods (MR-HF) in quantum dynamics [19]. In particular, it can be shown that the set of Tucker tensors of fixed rank forms an embedded finite-dimensional manifold [15]. Then the numerical treatment of this class of problems follows the general concepts of differential equations on manifolds [10]. Recently, similar results have been obtained for the TT format [13] and the HT format [23].

Some natural questions arise in the framework of topological tensor spaces. The first one is: is it possible to introduce a class of tensors containing Tucker, HT (and hence the TT) tensors with fixed and bounded rank ? A second question is: if such a class exists, is it possible to construct a parametrisation for the set of tensors of fixed rank in order to show that it is a true manifold even in infinite dimension? Finally, if the answers to both questions are yes, we would like to ask the following question: is the set of tensors of fixed rank an embedded submanifold of the topological tensor space, as ambient manifold, under consideration ?

The main goal of this paper is the study of the geometric structure of tensor representations based on subspaces. The paper is organised in two parts mainly. The first one, from Sect. 2 to Sect. 4, is devoted of preliminary definitions and results about, Banach spaces, Banach manifolds, Tensors spaces and the manifold of full rank tensors. Finally, from Sect. 5 to Sect. 8, we give the contributions of this paper. More precisely,

- In Sect. 5, we introduce a generalisation of the hierarchical tensor format in order to include the Tucker tensors (among others) in that class.
- In Sect. 6, we show that the set of tensors with fixed rank is an analytical Banach manifold and its geometric structure is independent on the ambient tensor Banach space under consideration.
- In Sect. 7, we show that when we have a tensor Hilbert space, as ambient space, the set of tensors with fixed rank is an embedded manifold whenever the norm of the ambient space is a uniform crossnorm (e.g. the classical L^2-norm and the Frobenius norm).
- In Sect. 8, we give a formalisation in this framework of the multi–configuration time–dependent Hartree MCTDH method (see [19]) in tensor Banach spaces.

2 Definitions and preliminary results

In the following, X is a Banach space with norm $\|\cdot\|$. The dual norm $\|\cdot\|_{X^*}$ of X^* is

$$\|\varphi\|_{X^*} = \sup\{|\varphi(x)| : x \in X \text{ with } \|x\|_X \leq 1\} = \sup\{|\varphi(x)| / \|x\|_X : 0 \neq x \in X\}. \tag{2.1}$$

By $\mathcal{L}(X,Y)$ we denote the space of continuous linear mappings from X into Y. The corresponding operator norm is written as $\|\cdot\|_{Y\leftarrow X}$.

Definition 2.1 *Let X be a Banach space. We say that $P \in \mathcal{L}(X,X)$ is a projection if $P \circ P = P$. In this situation we also say that P is a projection from X onto $P(X)$ parallel to* $\operatorname{Ker} P$.

From now on, we will denote $P \circ P = P^2$. Observe that if P is a projection then $I_X - P$ is also a projection. Moreover, $I_X - P$ is parallel to $P(X) := \operatorname{Im} P$.

Observe that each projection gives rise to a pair of subspaces, namely $U = \operatorname{Im} P$ and $V = \operatorname{Ker} P$ such that $X = U \oplus V$. It allows us to introduce the following two definitions.

Definition 2.2 *We will say that a subspace U of a Banach space X is a complemented subspace if U is closed and there exists V in X such that $X = U \oplus V$ and V is also a closed subspace of X. This subspace V is called a (topological) complement of U and (U,V) is a pair of complementary subspaces.*

Corresponding to each pair (U,V) of complementary subspaces, there is a projection P mapping X onto U along V, defined as follows. Since for each x there exists a unique decomposition $x = u + v$, where $u \in U$ and $v \in V$, we can define a linear map $P(u + v) := u$, where $\operatorname{Im} P = U$ and $\operatorname{Ker} P = V$. Moreover, $P^2 = P$.

Definition 2.3 *The* Grassmann manifold *of a Banach space X, denoted by $\mathbb{G}(X)$, is the set of all complemented subspaces of X.*

$U \in \mathbb{G}(X)$ holds if and only if U is a closed subspace and there exists a closed subspace V in X such that $X = U \oplus V$. Then, by the proof of Proposition 4.2 of [5], the following result can be shown.

Proposition 2.4 *Let X be a Banach space. The following conditions are equivalent:*

(a) $U \in \mathbb{G}(X)$.

(b) There exists $P \in \mathcal{L}(X,X)$ *such that* $P^2 = P$ *and* $\operatorname{Im} P = U$.

(c) There exists $Q \in \mathcal{L}(X,X)$ *such that* $Q^2 = Q$ *and* $\operatorname{Ker} Q = U$.

Let V and U be closed subspaces of a Banach space X such that $X = U \oplus V$. From now on, we will denote by $P_{U \oplus V}$ the projection onto U along V. Then we have $P_{V \oplus U} = I_X - P_{U \oplus V}$. Let $U, U' \in \mathbb{G}(X)$. We say that U and U' have a common complementary subspace in X, if $X = U \oplus W = U' \oplus W$ for some $W \in \mathbb{G}(X)$. The following result will be useful (see Lemma 2.1 in [3]).

Lemma 2.5 *Let X be a Banach space and assume that W, U, and U' are in $\mathbb{G}(X)$. Then the following statements are equivalent:*

(a) $X = U \oplus W = U' \oplus W$, *i.e., U and U' have a common complement in X.*

(b) $P_{U \oplus W}|_{U'} : U' \to U$ *has an inverse. Furthermore, if* $Q = \left(P_{U \oplus W}|_{U'}\right)^{-1}$ *exists, then Q is bounded and* $Q = P_{U' \oplus W}|_{U}$.

2.1 Banach manifolds

Definition 2.6 *Let $\mathbb{M}$ be a set. An atlas of class C^p $(p \geq 0)$ on $\mathbb{M}$ is a family of charts with some indexing set A, namely $\{(u_\alpha, M_\alpha) : \alpha \in A\}$, having the following properties:*

AT1 $\{M_\alpha\}_{\alpha \in A}$ *is a covering of* $\mathbb{M}$.

AT2 For each $\alpha \in A$, (u_α, M_α) stands for a bijection $u_\alpha : M_\alpha \to U_\alpha$ of M_α onto an open set U_α of a Banach space X_α, and for any α and β the set $u_\alpha(M_\alpha \cap M_\beta)$ is open in X_α.

AT3 Finally, if we let $M_\alpha \cap M_\beta = M_{\alpha\beta}$ and $u_\alpha(M_{\alpha\beta}) = U_{\alpha\beta}$, the transition mapping $u_\beta \circ u_\alpha^{-1} : U_{\alpha\beta} \to U_{\beta\alpha}$ is a C^p-diffeomorphism.

Two atlases are said *compatible* if each chart of one atlas is compatible with the other atlas. One verifies that the relation of compatibility between atlases is an equivalence relation.

Definition 2.7 *An equivalence class of atlases of class C^p on $\mathbb{M}$ is said to define a structure of a C^p-Banach manifold on $\mathbb{M}$, and hence we say that $\mathbb{M}$ is a Banach manifold. If X_α is a Hilbert space for all $\alpha \in A$, then we say that $\mathbb{M}$ is a Hilbert manifold.*

In condition AT2 we do not require that the Banach spaces are the same for all indices α, or even that they are isomorphic. If $X_\alpha = X$ for all α, we have the following definition.

Definition 2.8 *Let $\mathbb{M}$ be a set and X be a Banach space. We say that $\mathbb{M}$ is a C^p Banach manifold modelled on X if there exists an atlas of class C^p over $\mathbb{M}$ with $X_\alpha = X$ for all $\alpha \in A$.*

Example 2.9 *Every Banach space is a Banach manifold (for a Banach space Y, simply take (I_Y, Y) as atlas, where I_Y is the identity map on Y). In particular, the set of all bounded linear maps $\mathcal{L}(X,X)$ of a Banach space X is a Banach manifold.*

If X is a Banach space, then the set of all bounded linear automorphisms of X will be denoted by

$$\mathrm{GL}(X) := \{A \in \mathcal{L}(X,X) : A \text{ invertible }\}.$$

Example 2.10 *If X is a Banach space, then $\mathrm{GL}(X)$ is a Banach manifold, because it is an open set in $\mathcal{L}(X,X)$. Moreover, the map $A \mapsto A^{-1}$ is analytic (see 2.7 in [22]).*

Example 2.11 (Grassmann Banach manifold) *Let X be a Banach space. Then, following [4], it is possible to construct an atlas for $\mathbb{G}(X)$. To show that the atlas is an analytic Banach manifold, denote one of the complements of $U \in \mathbb{G}(X)$ by W, i.e., $X = U \oplus W$. Then we define the* Banach Grassmannian of U relative to W *by*

$$\mathbb{G}(W, X) := \{V \in \mathbb{G}(X) : X = V \oplus W\}.$$

It is possible to introduce a bijection

$$\Psi_{U\oplus W} : \mathbb{G}(W, X) \longrightarrow \mathcal{L}(U, W)$$

as the inverse of

$$\Psi_{U\oplus W}^{-1} : \mathcal{L}(U, W) \longrightarrow \mathbb{G}(W, X),$$

defined by

$$\Psi_{U\oplus W}^{-1}(L) = G(L) := \{u + L(u) : u \in U\}.$$

Observe that $\Psi_{U\oplus W}^{-1}(0) = U$ and $G(L) \oplus W = X$ for all $L \in \mathcal{L}(U, W)$. It can be shown that the collection $\{\Psi_{U\oplus W}, \mathbb{G}(W, X)\}_{U\in\mathbb{G}(X)}$ is an analytic atlas, and therefore, $\mathbb{G}(X)$ is an analytic Banach manifold. In particular, for each $U \in \mathbb{G}(X)$ the set $\mathbb{G}(W, X) \overset{\Psi_{U\oplus W}}{\cong} \mathcal{L}(U, W)$ is also a Banach manifold.

Let $\mathbb{M}$ be a Banach manifold of class $\mathcal{C}^p$, $p \geq 1$. Let m be a point of $\mathbb{M}$. We consider triples (U, φ, v) where (U, φ) is a chart at m and v is an element of the vector space in which $\varphi(U)$ lies. We say that two of such triples (U, φ, v) and (V, ψ, w) are *equivalent* if the derivative of $\psi\varphi^{-1}$ at $\varphi(m)$ maps v on w. Thanks to the chain rule it is an equivalence relation. An equivalence class of such triples is called a *tangent vector of* $\mathbb{M}$ *at* m.

Definition 2.12 *The set of such tangent vectors is called* tangent space of $\mathbb{M}$ at m *and it is denoted by* $\mathbb{T}_m(\mathbb{M})$.

Each chart (U, φ) determines a bijection of $\mathbb{T}_m(\mathbb{M})$ on a Banach space, namely the equivalence class of (U, φ, v) corresponds to the vector v. By means of such a bijection it is possible to equip $\mathbb{T}_m(\mathbb{M})$ with the structure of a topological vector space given by the chart, and it is immediate that this structure is independent of the chart selected.

Example 2.13 *If X is a Banach space, then $\mathbb{T}_x(X) = X$ for all $x \in X$.*

Example 2.14 *Let X be a Banach space and take $A \in \mathrm{GL}(X)$. Then $\mathbb{T}_A(\mathrm{GL}(X)) = \mathcal{L}(X, X)$.*

Example 2.15 *For $U \in \mathbb{G}(X)$ we have $\mathbb{T}_U(\mathbb{G}(X)) = \mathcal{L}(U, W)$.*

Example 2.16 *We point out that for a Hilbert space X with associated inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$, its unit sphere denoted by*

$$\mathbb{S}_X := \{x \in X : \|x\| = 1\},$$

is a Hilbert manifold of codimension one. Moreover, for each $x \in \mathbb{S}_X$, its tangent space is

$$\mathbb{T}_x(\mathbb{S}_X) = \mathrm{span}\,\{x\}^\perp = \{x' \in X : \langle x, x'\rangle = 0\}.$$

3 Tensor spaces

Concerning the definition of the algebraic tensor space ${}_a\bigotimes_{j=1}^d V_j$ generated from vector spaces V_j $(1 \leq j \leq d)$, we refer to Greub [8]. As underlying field we choose $\mathbb{R}$, but the results hold also for $\mathbb{C}$. The suffix 'a' in ${}_a\bigotimes_{j=1}^d V_j$ refers to the 'algebraic' nature. By definition, all elements of

$$\mathbf{V} := {}_a\bigotimes_{j=1}^{d} V_j$$

are *finite* linear combinations of elementary tensors $\mathbf{v} = \bigotimes_{j=1}^{d} v^{(j)} \; \left(v^{(j)} \in V_j\right)$.

The following notations and definitions will be useful. We recall that $L(V, W)$ is the space of linear maps from V into W, while $V' = L(V, \mathbb{R})$ is the algebraic dual of V. For metric spaces, $\mathcal{L}(V, W)$ denotes the continuous linear maps, while $V^* = \mathcal{L}(V, \mathbb{R})$ is the topological dual of V.

Let $D := \{1, \ldots, d\}$ be the index set of the 'spatial directions'. In the sequel, the index sets $D \backslash \{j\}$ will appear. Here, we use the abbreviations

$$\mathbf{V}_{[j]} := {}_a\bigotimes_{k \neq j} V_k \,, \qquad \text{where } \bigotimes_{k \neq j} \text{ means } \bigotimes_{k \in D \backslash \{j\}} . \tag{3.1}$$

Similarly, elementary tensors $\bigotimes_{k \neq j} v^{(k)}$ are denoted by $\mathbf{v}^{[j]}$.

For vector spaces V_j and W_j over $\mathbb{R}$, let linear mappings $A_j : V_j \to W_j$ $(1 \leq j \leq d)$ be given. Then the definition of the elementary tensor

$$\mathbf{A} = \bigotimes_{j=1}^{d} A_j : \; \mathbf{V} = {}_a\bigotimes_{j=1}^{d} V_j \longrightarrow \mathbf{W} = {}_a\bigotimes_{j=1}^{d} W_j$$

is given by

$$\mathbf{A}\left(\bigotimes_{j=1}^{d} v^{(j)}\right) := \bigotimes_{j=1}^{d} \left(A_j v^{(j)}\right). \tag{3.2}$$

Note that (3.2) extends uniquely to a linear mapping $\mathbf{A} : \mathbf{V} \to \mathbf{W}$.

Remark 3.1 *(a) Let* $\mathbf{V} := {}_a\bigotimes_{j=1}^{d} V_j$ *and* $\mathbf{W} := {}_a\bigotimes_{j=1}^{d} W_j$*. Then the linear combinations of tensor products of linear mappings* $\mathbf{A} = \bigotimes_{j=1}^{d} A_j$ *defined by means of (3.2) form a subspace of* $L(\mathbf{V}, \mathbf{W})$*:*

$${}_a\bigotimes_{j=1}^{d} L(V_j, W_j) \subset L(\mathbf{V}, \mathbf{W}).$$

(b) The special case of $W_j = \mathbb{R}$ *for all* j *(implying* $\mathbf{W} = \mathbb{R}$*) reads as* ${}_a\bigotimes_{j=1}^{d} V_j' \subset \mathbf{V}'$.
(c) If $\dim(V_j) < \infty$ *and* $\dim(W_j) < \infty$ *for all* j*, the inclusion '*$\subset$*' in (a) and (b) can be replaced by '*$=$*'. This can be easily verified by just checking the dimensions of spaces involved.*

Often, mappings $\mathbf{A} = \bigotimes_{j=1}^{d} A_j$ will appear, where most of the A_j are the identity (and therefore $V_j = W_j$). If $A_k \in L(V_k, W_k)$ for one k and $A_j = id$ for $j \neq k$, we use the following notation:

$$\mathbf{id}_{[k]} \otimes A_k := \underbrace{id \otimes \ldots \otimes id}_{k-1 \text{ factors}} \otimes A_k \otimes \underbrace{id \otimes \ldots \otimes id}_{d-k \text{ factors}} \in L(\mathbf{V}, \mathbf{V}_{[k]} \otimes_a W_k), \tag{3.3a}$$

provided that it is obvious which component k is meant. By the multiplication rule $\left(\bigotimes_{j=1}^{d} A_j\right) \circ \left(\bigotimes_{j=1}^{d} B_j\right) = \bigotimes_{j=1}^{d} (A_j \circ B_j)$ and since $id \circ A_j = A_j \circ id$, the following identity[1] holds for $j \neq k$:

$$\begin{aligned} & id \otimes \ldots \otimes id \otimes A_j \otimes id \otimes \ldots \otimes id \otimes A_k \otimes id \otimes \ldots \otimes id \\ & = (\mathbf{id}_{[j]} \otimes A_j) \circ (\mathbf{id}_{[k]} \otimes A_k) \\ & = (\mathbf{id}_{[k]} \otimes A_k) \circ (\mathbf{id}_{[j]} \otimes A_j) \end{aligned} \tag{3.3b}$$

(in the first line we assume $j < k$). Proceeding inductively with this argument over all indices, we obtain

$$\mathbf{A} = \bigotimes_{j=1}^{d} A_j = (\mathbf{id}_{[1]} \otimes A_1) \circ \cdots \circ (\mathbf{id}_{[d]} \otimes A_d).$$

[1] Note that the meaning of $\mathbf{id}_{[j]}$ and $\mathbf{id}_{[k]}$ may differ: in the second line of (3.3b), $(\mathbf{id}_{[k]} \otimes A_k) \in L(\mathbf{V}, \mathbf{V}_{[k]} \otimes_a W_k)$ and $(\mathbf{id}_{[j]} \otimes A_j) \in L\left(\mathbf{V}_{[k]} \otimes_a W_k, \mathbf{V}_{[j,k]} \otimes_a W_j \otimes_a W_k\right)$, whereas in the third one $(\mathbf{id}_{[j]} \otimes A_j) \in L(\mathbf{V}, \mathbf{V}_{[j]} \otimes_a W_j)$ and $(\mathbf{id}_{[k]} \otimes A_k) \in L\left(\mathbf{V}_{[j]} \otimes_a W_j, \mathbf{V}_{[j,k]} \otimes_a W_j \otimes_a W_k\right)$. Here $\mathbf{V}_{[j,k]} = {}_a\bigotimes_{l \in D \backslash \{j,k\}} V_l$.

If $W_j = \mathbb{R}$, i.e., if $A_j = \varphi_j \in V_j'$ is a linear form, then $\mathbf{id}_{[j]} \otimes \varphi_j \in L(\mathbf{V}, \mathbf{V}_{[j]})$ is used as symbol for $id \otimes \ldots \otimes id \otimes \varphi_j \otimes id \otimes \ldots \otimes id$ defined by

$$(\mathbf{id}_{[j]} \otimes \varphi_j) \left(\bigotimes_{k=1}^{d} v^{(k)} \right) = \varphi_j(v^{(j)}) \cdot \bigotimes_{k \neq j} v^{(k)}. \tag{3.3c}$$

Thus, if $\boldsymbol{\varphi} = \otimes_{j=1}^{d} \varphi_j \in \bigotimes_{j=1}^{d} V_j'$, we can also write

$$\boldsymbol{\varphi} = \otimes_{j=1}^{d} \varphi_j = (\mathbf{id}_{[1]} \otimes \varphi_1) \circ \cdots \circ (\mathbf{id}_{[d]} \otimes \varphi_d). \tag{3.3d}$$

Consider again the splitting of $\mathbf{V} = {}_a\bigotimes_{j=1}^{d} V_j$ into $\mathbf{V} = V_j \otimes_a \mathbf{V}_{[j]}$ with $\mathbf{V}_{[j]} := {}_a\bigotimes_{k \neq j} V_k$. For a linear form $\boldsymbol{\varphi}_{[j]} \in \mathbf{V}_{[j]}'$, the notation $id_j \otimes \boldsymbol{\varphi}_{[j]} \in L(\mathbf{V}, V_j)$ is used for the mapping

$$(id_j \otimes \boldsymbol{\varphi}_{[j]}) \left(\bigotimes_{k=1}^{d} v^{(k)} \right) = \boldsymbol{\varphi}_{[j]} \left(\bigotimes_{k \neq j} v^{(k)} \right) \cdot v^{(j)}. \tag{3.3e}$$

If $\boldsymbol{\varphi}_{[j]} = \bigotimes_{k \neq j} \varphi_k \in {}_a\bigotimes_{k \neq j} V_k'$ is an elementary tensor[2], $\boldsymbol{\varphi}_{[j]} \left(\bigotimes_{k \neq j} v^{(k)} \right) = \prod_{k \neq j} \varphi_k \left(v^{(k)} \right)$ holds in (3.3e). Finally, we can write (3.3d) as

$$\boldsymbol{\varphi} = \otimes_{j=1}^{d} \varphi_j = \varphi_j \circ (id_j \otimes \boldsymbol{\varphi}_{[j]}) \qquad \text{for } 1 \leq j \leq d. \tag{3.3f}$$

Definition 3.2 *We say that $\mathbf{V}_{\|\cdot\|}$ is a* Banach tensor space *if there exists an algebraic tensor space $\mathbf{V}$ and a norm $\|\cdot\|$ on $\mathbf{V}$ such that $\mathbf{V}_{\|\cdot\|}$ is the completion of $\mathbf{V}$ with respect to the norm $\|\cdot\|$, i.e.,*

$$\mathbf{V}_{\|\cdot\|} := {}_{\|\cdot\|}\bigotimes_{j=1}^{d} V_j = \overline{{}_a\bigotimes_{j=1}^{d} V_j}^{\|\cdot\|}.$$

If $\mathbf{V}_{\|\cdot\|}$ is a Hilbert space, we say that $\mathbf{V}_{\|\cdot\|}$ is a Hilbert tensor space.

Next, we give some examples of Banach and Hilbert tensor spaces.

Example 3.3 *For $I_j \subset \mathbb{R}$ $(1 \leq j \leq d)$ and $1 \leq p < \infty$, the Sobolev space $H^{N,p}(I_j)$ consists of all univariate functions f from $L^p(I_j)$ with bounded norm*[3]

$$\|f\|_{N,p;I_j} := \left(\sum_{n=0}^{N} \int_{I_j} |\partial^n f|^p \, \mathrm{d}x \right)^{1/p}, \tag{3.4a}$$

whereas the space $H^{N,p}(\mathbf{I})$ of d-variate functions on $\mathbf{I} = I_1 \times I_2 \times \ldots \times I_d \subset \mathbb{R}^d$ is endowed with the norm

$$\|f\|_{N,p} := \Big(\sum_{0 \leq |\mathbf{n}| \leq N} \int_{\mathbf{I}} |\partial^{\mathbf{n}} f|^p \, \mathrm{d}\mathbf{x} \Big)^{1/p} \tag{3.4b}$$

with $\mathbf{n} \in \mathbb{N}_0^d$ being a multi-index of length $|\mathbf{n}| := \sum_{j=1}^{d} n_j$. For $p > 1$ it is well-known that $H^{N,p}(I_j)$ and $H^{N,p}(\mathbf{I})$ are reflexive and separable Banach spaces. Moreover, for $p = 2$, the Sobolev spaces $H^N(I_j) := H^{N,2}(I_j)$ and $H^N(\mathbf{I}) := H^{N,2}(\mathbf{I})$ are Hilbert spaces. As a first example,

$$H^{N,p}(\mathbf{I}) = {}_{\|\cdot\|_{N,p}}\bigotimes_{j=1}^{d} H^{N,p}(I_j)$$

is a Banach tensor space. Examples of Hilbert tensor spaces are

$$L^2(\mathbf{I}) = {}_{\|\cdot\|_{0,2}}\bigotimes_{j=1}^{d} L^2(I_j) \quad \text{and} \quad H^N(\mathbf{I}) = {}_{\|\cdot\|_{N,2}}\bigotimes_{j=1}^{d} H^N(I_j) \ \text{ for } N \in \mathbb{N}.$$

[2] Recall that an elementary tensor is a tensor of the form $v_1 \otimes \cdots \otimes v_d$.

[3] It suffices to have in (3.4a) the terms $n = 0$ and $n = N$. The derivatives are to be understood as weak derivatives.

Let $\|\cdot\|_j$, $1 \le j \le d$, be the norms of the vector spaces V_j appearing in $\mathbf{V} = {}_a\bigotimes_{j=1}^d V_j$. By $\|\cdot\|$ we denote the norm on the tensor space $\mathbf{V}$. Note that $\|\cdot\|$ is not determined by $\|\cdot\|_j$, but there are relations which are 'reasonable'.

Any norm $\|\cdot\|$ on ${}_a\bigotimes_{j=1}^d V_j$ satisfying

$$\left\|\bigotimes_{j=1}^d v^{(j)}\right\| = \prod_{j=1}^d \|v^{(j)}\|_j \qquad \text{for all } v^{(j)} \in V_j \;\; (1 \le j \le d) \tag{3.5}$$

is called a *crossnorm*. As usual, the dual norm to $\|\cdot\|$ is denoted by $\|\cdot\|^*$. If $\|\cdot\|$ is a crossnorm and also $\|\cdot\|^*$ is a crossnorm on ${}_a\bigotimes_{j=1}^d V_j^*$, i.e.,

$$\left\|\bigotimes_{j=1}^d \varphi^{(j)}\right\|^* = \prod_{j=1}^d \|\varphi^{(j)}\|_j^* \qquad \text{for all } \varphi^{(j)} \in V_j^* \;\; (1 \le j \le d)\,, \tag{3.6}$$

$\|\cdot\|$ is called a *reasonable crossnorm.*

Remark 3.4 *Eq. (3.5) implies the inequality $\|\bigotimes_{j=1}^d v_j\| \lesssim \prod_{j=1}^d \|v_j\|_j$ which is equivalent to the continuity of the multilinear tensor product mapping*[4]

$$\bigotimes : \bigtimes_{j=1}^d \left(V_j, \|\cdot\|_j\right) \longrightarrow \left({}_a\bigotimes_{j=1}^d V_j\ , \|\cdot\|\right), \tag{3.7}$$

defined by $\otimes((v_1, \ldots, v_d)) = \otimes_{j=1}^d v_j$, the product space being equipped with the product topology induced by the maximum norm $\|(v_1, \ldots, v_d)\| = \max_{1\le j\le d} \|v_j\|_j$.

Proposition 3.5 *Assume that the tensor product map (3.7) is continuous. Then it is also Fréchet differentiable and its differential is given by*

$$D\left(\bigotimes(v_1, \ldots, v_d)\right)(w_1, \ldots, w_d) = \sum_{j=1}^d v_1 \otimes \ldots \otimes v_{j-1} \otimes w_j \otimes v_{j+1} \otimes \cdots v_d.$$

Proof. Clearly, $D\bigotimes(v_1, \ldots, v_d)$ is a multilinear map. If we assume that the tensor product map (3.7) is continuous, that is $\|\bigotimes_{j=1}^d u_j\| \le C \prod_{j=1}^d \|u_j\|_j$ for some $C > 0$, then

$$\begin{aligned}\|D\bigotimes(v_1, \ldots, v_d)(w_1, \ldots, w_d)\| &\le C \sum_{j=1}^d \|v_1\|_1 \cdots \|v_{j-1}\|_{j-1} \|w_j\|_j \|v_{j+1}\|_{j+1} \cdots \|v_d\|_d \\ &\le C \left(\sum_{j=1}^d \frac{\prod_{k=1}^d \|v_k\|_k}{\|v_j\|_j}\right) \max_{1\le k\le d} \|w_k\|_k\end{aligned}$$

shows that $D\bigotimes(v_1, \ldots, v_d)$ is also continuous. Finally,

$$\begin{aligned}&\|\otimes(v_1 + h_1, \cdots, v_d + h_d) - \otimes(v_1, \cdots, v_d) - D\otimes(v_1, \cdots, v_d)(h_1, \cdots, h_d)\| \\ &= \sum_{\substack{i_1,i_2=1\\ i_1<i_2}}^d \|T_{i_1,i_2}(h_{i_1}, h_{i_2}) + \sum_{\substack{i_1,i_2,i_3=1\\ i_1<i_2<i_3}}^d T_{i_1,i_2,i_3}(h_{i_1}, h_{i_2}, h_{i_3}) + \ldots + T_{1,\ldots,d}(h_1, \ldots, h_d)\| \\ &\le \|\sum_{\substack{i_1,i_2=1\\ i_1<i_2}}^d T_{i_1,i_2}(h_{i_1}, h_{i_2})\| + \sum_{\substack{i_1,i_2,i_3=1\\ i_1<i_2<i_3}}^d \|T_{i_1,i_2,i_3}(h_{i_1}, h_{i_2}, h_{i_3})\| + \ldots + \|T_{1,\ldots,d}(h_1, \ldots, h_d)\|\end{aligned}$$

---

[4]Recall that a multilinear map T from $\bigtimes_{j=1}^d (V_j, \|\cdot\|_j)$ equipped with the product topology to a normed space $(W, \|\cdot\|)$ is continuous if and only if $\|T\| < \infty$, with

$$\|T\| := \sup_{\substack{(v_1,\ldots,v_d)\\ \|(v_1,\ldots,v_d)\|\le 1}} \|T(v_1, \ldots, v_d)\| = \sup_{\substack{(v_1,\ldots,v_d)\\ \|v_1\|_1\le 1,\ldots,\|v_d\|_d\le 1}} \|T(v_1, \ldots, v_d)\| = \sup_{(v_1,\ldots,v_d)} \frac{\|T(v_1, \ldots, v_d)\|}{\|v_1\|_1 \cdots \|v_d\|_d}.$$

where the $T_{i_1,\ldots,i_k}$ are multilinear maps defined by $T_{i_1,\ldots,i_k}(h_{i_1},\ldots,h_{i_k}) = \otimes_{j=1}^d z_j$ with $z_j = h_j$ if $j \in \{i_1,\ldots,i_k\}$, and $z_j = v_j$ otherwise. Since these multilinear maps have at least two arguments, we have

$$\begin{aligned}
\|T_{i_1,\ldots,i_k}(h_{i_1},\ldots,h_{i_k})\| &\leq C \prod_{j\in\{i_1,\ldots,i_k\}} \|h_j\|_j \prod_{j\in\{1,\ldots,d\}\setminus\{i_1,\ldots,i_k\}} \|v_j\|_j \\
&\leq C \max_{1\leq j\leq d} \|h_j\|_j \prod_{j\in\{i_2,\ldots,i_k\}} \|h_j\|_j \prod_{j\in\{1,\ldots,d\}\setminus\{i_1,\ldots,i_k\}} \|v_j\|_j \\
&= C\|(h_1,\ldots,h_d)\| \prod_{j\in\{i_2,\ldots,i_k\}} \|h_j\|_j \prod_{j\in\{1,\ldots,d\}\setminus\{i_1,\ldots,i_k\}} \|v_j\|_j
\end{aligned}$$

which proves that $\frac{\|T_{i_1,\ldots,i_k}(h_{i_1},\ldots,h_{i_k})\|}{\|(h_1,\ldots,h_d)\|}$ tends to zero as $(h_1,\ldots,h_d)\to 0$, and therefore $\bigotimes$ is Fréchet differentiable and the proposition follows. ■

Grothendieck [9] named the following norm $\|\cdot\|_\vee$ the *injective norm.*

Definition 3.6 *Let V_i be a Banach space with norm $\|\cdot\|_i$ for $1 \leq i \leq d$. Then for $\mathbf{v} \in \mathbf{V} = {}_a\bigotimes_{j=1}^d V_j$ define $\|\cdot\|_\vee$ by*

$$\|\mathbf{v}\|_\vee := \sup\left\{ \frac{|(\varphi_1\otimes\varphi_2\otimes\ldots\otimes\varphi_d)(\mathbf{v})|}{\prod_{j=1}^d \|\varphi_j\|_j^*} : 0 \neq \varphi_j \in V_j^*, 1\leq j\leq d \right\}. \tag{3.8}$$

It is well-known that the injective norm is a reasonable crossnorm (see Lemma 1.6 in [17]). Further properties are given by the next proposition (see also 4.2.4 in [12]).

Proposition 3.7 *The following statements hold.*

(a) The injective norm is the weakest reasonable crossnorm on $\mathbf{V}$, i.e., if $\|\cdot\|$ is a reasonable crossnorm on $\mathbf{V}$, then

$$\|\cdot\| \gtrsim \|\cdot\|_\vee \tag{3.9}$$

(b) For any norm $\|\cdot\|$ on $\mathbf{V}$ satisfying $\|\cdot\|_\vee \lesssim \|\cdot\|$, the map (3.7) is continuous, and hence Fréchet differentiable.

Below, we will need a further assumption on the norm $\|\cdot\|$. A norm $\|\cdot\|$ is a *uniform crossnorm* if it is a crossnorm (cf. (3.5)) and satisfies

$$\left\| \left(\bigotimes_{j=1}^d A_j \right)(\mathbf{v}) \right\| \leq \left(\prod_{j=1}^d \|A_j\|_{V_j\leftarrow V_j} \right) \|\mathbf{v}\| \tag{3.10}$$

for all $A_j \in \mathcal{L}(V_j,V_j)$ $(1\leq j\leq d)$ and all $\mathbf{v}\in {}_a\bigotimes_{j=1}^d V_j$. The uniform crossnorm property implies that $\|\cdot\|$ is a reasonable crossnorm (cf. [20]). Hence, condition (3.9) is ensured (cf. Proposition 3.7a).

Definition 3.8 *Let X be a Banach space and $\|\cdot\|$ be a norm defined over ${}_a\bigotimes_{j=1}^d V_j$. For each $A \in \mathcal{L}\left({}_a\bigotimes_{j=1}^d V_j\ , X\right)$ we will denote by $\overline{A} \in \mathcal{L}\left({}_{\|\cdot\|}\bigotimes_{j=1}^d V_j\ , X\right)$ its unique extension. Recall that $\overline{A}|_{\mathbf{V}} = A$.*

Observe that if $\|\cdot\|$ is a uniform crossnorm then for all $A_j \in \mathcal{L}(V_j,V_j)$ $(1\leq j\leq d)$ the map $\overline{\bigotimes_{j=1}^d A_j}$ belongs to $\mathcal{L}(\mathbf{V}_{\|\cdot\|},\mathbf{V}_{\|\cdot\|})$.

4 The manifold of multilinear full rank tensors

Now, we assume that $\dim V_k < \infty$ for $1\leq k\leq d$. Before introducing the set of multilinear full rank tensors, we recall the definition of the 'matricisation' (or 'unfolding') of a tensor in a finite-dimensional setting.

Definition 4.1 *For $j \in D = \{1, \ldots, d\}$, the map $\mathcal{M}_j$ is defined as the isomorphism*

$$\begin{array}{rccl} \mathcal{M}_j : & {}_a\bigotimes_{s=1}^d V_s & \to & V_j \otimes_a V_{[j]}, \\ & \bigotimes_{s=1}^d v^{(s)} & \mapsto & v^{(j)} \otimes \mathbf{v}^{[j]} \text{ with } \mathbf{v}^{[j]} := \bigotimes_{k \neq j} v^{(k)}. \end{array}$$

In the finite-dimensional case of $V_k = \mathbb{R}^{n_k}$, an element of the tensor space $V_j \otimes_a V_{[j]}$ of order 2 may be considered as a matrix from $\mathbb{R}^{n_j \times n_{[j]}}$, where $n_{[j]} = \prod_{k \neq j} n_k$. Then, $\mathcal{M}_j$ maps a tensor entry $\mathbf{v}[i_1, \ldots, i_j, \ldots, i_d]$ into the matrix entry $(\mathcal{M}_j(\mathbf{v}))\,[i_j, (i_1, \ldots, i_{j-1}, i_{j+1}, \ldots, i_d)]$. As long as we do not consider matrix properties which depend on the ordering of the index set, we need not introduce an ordering of the $(d-1)$-tuple $(i_1, \ldots, i_{j-1}, i_{j+1}, \ldots, i_d)$.

Next, we restrict the considerations to finite-dimensional V_k. Since tensor products of two vectors can be interpreted as matrices, the mapping $\mathcal{M}_j$ is named 'matricisation' (or 'unfolding'). The interpretation of tensors $\mathbf{v}$ as matrices enables us to transfer the matrix terminology to $\mathbf{v}$. In particular, we may define the rank of $\mathcal{M}_j(\mathbf{v})$ as a property of $\mathbf{v}$.

Definition 4.2 *Let* $\dim(V_k) < \infty$ $(k \in D)$. *For all $j \in D$ we define*

$$\operatorname{rank}_j(\mathbf{v}) := \operatorname{rank}(\mathcal{M}_j(\mathbf{v})). \tag{4.1}$$

Assume $\dim(V_k) < \infty$ $(k \in D)$. Since $\mathcal{M}_j(\mathbf{v}) \in \mathbb{R}^{\dim(V_j) \times \prod_{k \neq j} \dim(V_k)}$, if[5] $\operatorname{rank}(\mathcal{M}_j(\mathbf{v})) = \dim(V_j)$ then

$$\det\left(\mathcal{M}_j(\mathbf{v})\mathcal{M}_j(\mathbf{v})^T\right) \neq 0.$$

It allows to introduce the following definition.

Definition 4.3 *Let $\mathbf{v} \in \mathbb{R}^{n_1 \times \cdots \times n_d}$. We say that $\mathbf{v}$ is a multilinear full rank tensor if and only if*

$$\prod_{j=1}^d \det\left(\mathcal{M}_j(\mathbf{v})\mathcal{M}_j(\mathbf{v})^T\right) \neq 0. \tag{4.2}$$

We denote by $\mathbb{R}_*^{n_1 \times \cdots \times n_d}$ the set of multilinear full rank tensors of $\mathbb{R}^{n_1 \times \cdots \times n_d}$. Since the determinant is a continuous function, $\mathbb{R}_*^{n_1 \times \cdots \times n_d}$ is an open set in $\mathbb{R}^{n_1 \times \cdots \times n_d}$, and hence a finite-dimensional manifold. Moreover, the tangent space $\mathbb{T}_{\mathbf{v}}(\mathbb{R}_*^{n_1 \times \cdots \times n_d}) = \mathbb{R}^{n_1 \times \cdots \times n_d}$ for all $\mathbf{v} \in \mathbb{R}_*^{n_1 \times \cdots \times n_d}$ (cf. Definition 2.12).

5 Minimal subspaces and the representation of tensors in the tree based format

We introduce the abbreviate TBF for 'tree based format'. For instance, a TBF tensor is a tensor represented in the tree based format, etc. The tensor based rank will be abbreviated by TB rank. The underlying tree will be defined in Sect. 5.2.

5.1 Minimal subspaces

Let V_j be a vector space for $j \in D$, where D is a finite index set, and consider a tensor space $\mathbf{V}_D := {}_a\bigotimes_{j \in D} V_j$. In order to avoid trivial cases, we assume $\#D \geq 2$.

Definition 5.1 *For a tensor $\mathbf{v} \in {}_a\bigotimes_{j=1}^d V_j$, the* minimal subspaces *are denoted by $U_j^{\min}(\mathbf{v})$ $(j = 1, \ldots, d)$ and defined by the property that $\mathbf{v} \in {}_a\bigotimes_{j=1}^d U_j$ implies $U_j^{\min}(\mathbf{v}) \subset U_j$ $(j = 1, \ldots, d)$, while $\mathbf{v} \in {}_a\bigotimes_{j=1}^d U_j^{\min}(\mathbf{v})$.*

A useful result is the following.

[5] $\operatorname{rank}(\mathcal{M}_j(\mathbf{v})) = \dim(V_j)$ implies that $\dim(V_j) \leq \prod_{k \neq j} \dim(V_k)$ and that $\mathcal{M}_j(\mathbf{v})$ has full rank. The latter estimate is a very natural assumption.

Lemma 5.2 *Let* $\mathbf{u},\mathbf{v} \in {}_a\bigotimes_{j=1}^d V_j$ *be such that* $\dim U_j^{\min}(\mathbf{v}) = \dim U_j^{\min}(\mathbf{u})$ *for* $1 \le j \le d$. *Then*

$$\mathbf{u} \in {}_a\bigotimes_{j=1}^{d} U_j^{\min}(\mathbf{v})$$

if and only if $U_j^{\min}(\mathbf{u}) = U_j^{\min}(\mathbf{v})$ *for* $j = 1,2,\ldots,d$.

Proof. Clearly, if $U_j^{\min}(\mathbf{u}) = U_j^{\min}(\mathbf{v})$ for $j = 1,2,\ldots,d$, then $\mathbf{u} \in {}_a\bigotimes_{j=1}^d U_j^{\min}(\mathbf{v})$ holds. On the other hand assume that $\mathbf{u} \in {}_a\bigotimes_{j=1}^d U_j^{\min}(\mathbf{v})$. We have

$$U_j^{\min}(\mathbf{u}) \subset U_j^{\min}(\mathbf{v}) \text{ for } 1 \le j \le d.$$

Since $\dim U_j^{\min}(\mathbf{v}) = \dim U_j^{\min}(\mathbf{u})$ for $1 \le j \le d$, we obtain the desired equality and the lemma follows. ∎

The next characterisation of $U_j^{\min}(\mathbf{v})$ is due to [6]. To this end, we introduce the following two subspaces:

$$U_j^{I}(\mathbf{v}) := \left\{(id_j \otimes \boldsymbol{\varphi}_{[j]})(\mathbf{v}) : \; \boldsymbol{\varphi}_{[j]} \in {}_a\bigotimes\nolimits_{k\neq j} V_k'\right\}, \tag{5.1a}$$

$$U_j^{II}(\mathbf{v}) := \left\{(id_j \otimes \boldsymbol{\varphi}_{[j]})(\mathbf{v}) : \; \boldsymbol{\varphi}_{[j]} \in \left({}_a\bigotimes\nolimits_{k\neq j} V_k\right)'\right\}. \tag{5.1b}$$

In the case of normed spaces V_k, we may consider the subspace

$$U_j^{III}(\mathbf{v}) := \left\{(id_j \otimes \boldsymbol{\varphi}_{[j]})(\mathbf{v}) : \; \boldsymbol{\varphi}_{[j]} \in {}_a\bigotimes\nolimits_{k\neq j} V_k^*\right\}. \tag{5.1c}$$

Finally, if $\mathbf{V}_{[j]} = {}_a\bigotimes_{k\neq j} V_k$ is a normed space, we can define

$$U_j^{IV}(\mathbf{v}) := \left\{(id_j \otimes \boldsymbol{\varphi}_{[j]})(\mathbf{v}) : \; \boldsymbol{\varphi}_{[j]} \in \mathbf{V}_{[j]}^*\right\}. \tag{5.1d}$$

Note that, in general, the four spaces ${}_a\bigotimes_{k\neq j} V_k'$, $({}_a\bigotimes_{k\neq j} V_k)'$, ${}_a\bigotimes_{k\neq j} V_k^*$ and $\mathbf{V}_{[j]}^*$ may differ.

Theorem 5.3 *For any* $\mathbf{v} \in \mathbf{V} = {}_a\bigotimes_{j=1}^d V_j$, *the following statements hold:*

(a) There exist minimal subspaces $U_j^{\min}(\mathbf{v})$ $(1 \le j \le d)$, *whose algebraic characterisation is given by*

$$U_j^{\min}(\mathbf{v}) = U_j^{I}(\mathbf{v}) = U_j^{II}(\mathbf{v}).$$

(b) Assume that V_j *and* $\mathbf{V}_{[j]} = {}_a\bigotimes_{k\neq j} V_k$ *are normed spaces for* $1 \le j \le d$. *Then*

$$U_j^{\min}(\mathbf{v}) = U_j^{I}(\mathbf{v}) = U_j^{II}(\mathbf{v}) = U_j^{III}(\mathbf{v}) = U_j^{IV}(\mathbf{v}).$$

(c) If $\dim V_j < \infty$ *for* $j \in D$, *then*

$$\dim U_j^{\min}(\mathbf{v}) = \mathrm{rank}_j(\mathbf{v}).$$

The minimal subspaces are useful to introduce the following sets of tensor representations based on subspaces. Fix $\mathbf{r} = (r_1,\ldots,r_d) \in \mathbb{N}^d$. Then we define *the set of Tucker tensors with bounded rank* $\mathbf{r}$ *in* $\mathbf{V} = {}_a\bigotimes_{j=1}^d V_j$ by

$$\mathcal{T}_{\mathbf{r}}(\mathbf{V}) := \left\{\mathbf{v} \in \mathbf{V} : \dim U_j^{\min}(\mathbf{v}) \le r_j,\, 1 \le j \le d\right\},$$

and *the set of Tucker tensors with fixed rank* $\mathbf{r}$ *in* $\mathbf{V} = {}_a\bigotimes_{j=1}^d V_j$ by

$$\mathcal{M}_{\mathbf{r}}(\mathbf{V}) := \left\{\mathbf{v} \in \mathbf{V} : \dim U_j^{\min}(\mathbf{v}) = r_j,\, 1 \le j \le d\right\}.$$

Then $\mathcal{M}_{\mathbf{r}}(\mathbf{V}) \subset \mathcal{T}_{\mathbf{r}}(\mathbf{V}) \subset \mathbf{V}$ holds.

We have introduced the minimal subspace $U_j^{\min}(\mathbf{v}) \subset V_j$ for a singleton $\{j\} \subset D := \{1, 2, \ldots, d\}$. Instead we may consider general disjoint and non-empty subsets $\alpha_i \subset D$. For instance, let $\mathbf{v} \in {}_a\bigotimes_{j\in D} V_j = \mathbf{V}_{\alpha_1} \otimes \mathbf{V}_{\alpha_2} \otimes \mathbf{V}_{\alpha_3}$, where $\alpha_1 = \{1, 2\}$, $\alpha_2 = \{3, 4\}$, and $\alpha_3 = \{5, 6, 7\}$. Then we can conclude that there are minimal subspaces $\mathbf{U}_{\alpha_\nu}^{\min}(\mathbf{v})$ for $\nu = 1, 2, 3$, such that $\mathbf{v} \in {}_a\bigotimes_{\nu=1}^{3} \mathbf{U}_{\alpha_\nu}^{\min}(\mathbf{v})$. The relation between $U_j^{\min}(\mathbf{v})$ and $\mathbf{U}_{\alpha_\nu}^{\min}(\mathbf{v})$ is as follows.

Proposition 5.4 *Let $\mathbf{v} \in \mathbf{V}_D = {}_a\bigotimes_{j\in D} V_j$ and $\emptyset \neq \alpha \subset D$. Then the minimal subspaces $\mathbf{U}_\alpha^{\min}(\mathbf{v})$ and $U_j^{\min}(\mathbf{v})$ for $j \in \alpha$ are related by*

$$\mathbf{U}_\alpha^{\min}(\mathbf{v}) \subset {}_a\bigotimes_{j\in\alpha} U_j^{\min}(\mathbf{v}) . \tag{5.2}$$

An obvious generalisation of the previous result is given below.

Corollary 5.5 *Let $\mathbf{v} \in \mathbf{V}_D = {}_a\bigotimes_{j\in D} V_j$. Assume that $\emptyset \neq \alpha_i \subset D$ are pairwise disjoint for $i = 1, 2, \ldots, m$. The minimal subspace $\mathbf{U}_\alpha^{\min}(\mathbf{v})$ for $\alpha := \bigcup_{i=1}^m \alpha_i$ satisfies*

$$\mathbf{U}_\alpha^{\min}(\mathbf{v}) \subset {}_a\bigotimes_{i=1}^{m} \mathbf{U}_{\alpha_i}^{\min}(\mathbf{v}) . \tag{5.3}$$

The algebraic characterisation of $\mathbf{U}_\alpha^{\min}(\mathbf{v})$ is analogous to that given in Theorem 5.3. Formulae (5.1a,b) become

$$\begin{aligned}\mathbf{U}_\alpha^{\min}(\mathbf{v}) &= \left\{ (id_\alpha \otimes \boldsymbol{\varphi}_{\alpha^c})(\mathbf{v}) : \boldsymbol{\varphi}_{\alpha^c} \in {}_a\bigotimes\nolimits_{j\in\alpha^c} V_j' \right\} \\ &= \left\{ (id_\alpha \otimes \boldsymbol{\varphi}_{\alpha^c})(\mathbf{v}) : \boldsymbol{\varphi}_{\alpha^c} \in \left({}_a\bigotimes\nolimits_{j\in\alpha^c} V_j \right)' \right\},\end{aligned} \tag{5.4}$$

where $(id_\alpha \otimes \boldsymbol{\varphi}_{\alpha^c})(\otimes_{j=1}^d v^{(j)}) = \left(\boldsymbol{\varphi}_{\alpha^c}(\otimes_{j\in\alpha^c} v^{(j)})\right) \otimes_{k\in\alpha} v^{(k)}$. The analogues of (5.1c,d) apply as soon as norms are defined on V_j and ${}_a\bigotimes_{j\in\alpha^c} V_j$.

From now on, given $\emptyset \neq \alpha \subset D$, we will denote $\mathbf{V}_\alpha := {}_a\bigotimes_{j\in\alpha} V_j$, $r_\alpha := \dim U_\alpha^{\min}(\mathbf{v})$ and $U_D^{\min}(\mathbf{v}) := \operatorname{span}\{\mathbf{v}\}$.

Example 5.6 *Let us consider $D = \{1, 2, 3, 4, 5, 6\}$, then*

$$\mathbf{V}_D = {}_a\bigotimes_{j=1}^{6} V_j = \left({}_a\bigotimes_{j=1}^{3} V_j \right) \otimes_a \left({}_a\bigotimes_{j=4}^{5} V_j \right) \otimes_a V_6 = \mathbf{V}_{123} \otimes_a \mathbf{V}_{45} \otimes_a V_6.$$

It is well-known (see [6]) that $\mathbf{v} \in {}_a\bigotimes_{j=1}^{6} U_j^{\min}(\mathbf{v})$ and $\mathbf{v} \in U_{123}^{\min}(\mathbf{v}) \otimes_a U_{45}^{\min}(\mathbf{v}) \otimes_a U_6^{\min}(\mathbf{v})$. From Proposition 5.4 we have

$$U_D^{\min}(\mathbf{v}) \subset U_{123}^{\min}(\mathbf{v}) \otimes_a U_{45}^{\min}(\mathbf{v}) \otimes_a U_6^{\min}(\mathbf{v}) \subset {}_a\bigotimes_{j=1}^{6} U_j^{\min}(\mathbf{v}) . \tag{5.5}$$

Moreover, we can write

$$\mathbf{v} = \sum_{i_{123}=1}^{r_{123}} \sum_{i_{45}=1}^{r_{45}} \sum_{i_6=1}^{r_6} C^{(D)}_{i_{123} i_{45} i_6} \, \mathbf{u}^{(123)}_{i_{123}} \otimes \mathbf{u}^{(45)}_{i_{45}} \otimes u^{(6)}_{i_6}, \quad C^{(D)} \in \mathbb{R}_*^{r_{123}\times r_{45}\times r_6}$$

where

$$\mathbf{u}_{i_{123}} = \sum_{i_1=1}^{r_1} \sum_{i_2=1}^{r_2} \sum_{i_3=1}^{r_3} C^{(123)}_{i_{123}, i_1 i_2 i_3} \, u^{(1)}_{i_1} \otimes u^{(2)}_{i_2} \otimes u^{(3)}_{i_3}, \quad C^{(123)} \in \mathbb{R}_*^{r_{123}\times r_1 r_2 r_3}$$

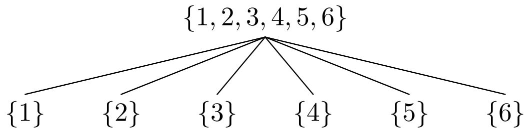

Figure 5.1: A dimension partition tree related to $U_D^{\min}(\mathbf{v}) \subset {}_a\bigotimes_{j\in D} U_j^{\min}(\mathbf{v})$.

and

$$\mathbf{u}_{i_{45}} = \sum_{i_4=1}^{r_4}\sum_{i_5=1}^{r_5} C^{(45)}_{i_{45},i_4 i_5}\, u^{(4)}_{i_4} \otimes u^{(5)}_{i_5}, \quad C^{(45)} \in \mathbb{R}_*^{r_{45}\times r_4 r_5}.$$

Finally

$$\mathbf{v} = \sum_{i_1=1}^{r_1}\cdots\sum_{i_6=1}^{r_6} \underbrace{\left(\sum_{i_{123}=1}^{r_{123}}\sum_{i_{45}=1}^{r_{45}} C^{(D)}_{i_{123}i_{45}i_6} C^{(123)}_{i_{123},i_1 i_2 i_3} C^{(45)}_{i_{45},i_4 i_5}\right)}_{v_{i_1,\ldots,i_6}} \bigotimes_{k=1}^{6} u^{(k)}_{i_k},$$

where $u^{(k)}_{i_k} \in U_k^{\min}(\mathbf{v})$ *for* $1 \le k \le 6$.

5.2 Dimension partition tree and TB rank

Since (5.5) can be represented by means of a tree (see Figure 5.2), it motivates the following definition.

Definition 5.7 *The tree* T_D *is called* a dimension partition tree of D *if*

(a) all vertices $\alpha \in T_D$ *are non–empty subsets of* D,

(b) D *is the root of* T_D,

(c) every vertex $\alpha \in T_D$ *with* $\#\alpha \ge 2$ *has at least two sons. Moreover, if* $S(\alpha) \subset 2^D$ *denotes the set of sons of* α *then* $\alpha = \cup_{\beta\in S(\alpha)}\beta$ *where* $\beta \cap \beta' = \emptyset$ *for all* $\beta, \beta' \in S(\alpha)$, $\beta \neq \beta'$.

If $S(\alpha) = \emptyset$, α is called a *leaf*. The set of leaves is denoted by $\mathcal{L}(T_D)$. An easy consequence of Definition 5.7 is that the set of leaves $\mathcal{L}(T_D)$ coincides with the singletons of D, i.e., $\mathcal{L}(T_D) = \{\{j\} : j \in D\}$.

Example 5.8 *Consider* $D = \{1,2,3,4,5,6\}$ *and recall that* $U_D^{\min}(\mathbf{v}) \subset {}_a\bigotimes_{j\in D} U_j^{\min}(\mathbf{v})$. *Take*

$$T_D = \{D, \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}\} \text{ and } S(D) = \{\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}\}$$

(see Figure 5.1). Then $S(D) = \mathcal{L}(T_D)$.

Example 5.9 *In Figure 5.2 we have a tree which corresponds to (5.5). Here* $D = \{1,2,3,4,5,6\}$ *and*

$$T_D = \{D, \{1,2,3\}, \{4,5\}, \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}\},$$
$$S(D) = \{\{1,2,3\}, \{4,5\}, \{6\}\},\ S(\{4,5\}) = \{\{4\}, \{5\}\},\ S(\{1,2,3\}) = \{\{1\}, \{2\}, \{3\}\}.$$

Moreover

$$\mathcal{L}(T_D) = \{\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}\}.$$

Observe that for each $\mathbf{v} \in \mathbf{V}_D$ we have that $(\dim \mathbf{U}_\alpha^{\min}(\mathbf{v}))_{\alpha \in 2^D\setminus\{\emptyset\}}$ is in $\mathbb{N}^{2^{\#D}-1}$,

Definition 5.10 *Let* T_D *be a dimension partition tree of an index set* D. *Then for each* $\mathbf{v} \in \mathbf{V}_D = {}_a\bigotimes_{j\in D} V_j$ *we define its* tensor based rank (TB rank) *by* $(\dim \mathbf{U}_\alpha^{\min}(\mathbf{v}))_{\alpha\in T_D} \in \mathbb{N}^{\#T_D}$.

In order to characterise the tensors $\mathbf{v} \in \mathbf{V}_D$ satisfying $(\dim \mathbf{U}_\alpha^{\min}(\mathbf{v}))_{\alpha\in T_D} = \mathfrak{r}$, for a fixed $\mathfrak{r} := (r_\alpha)_{\alpha\in T_D} \in \mathbb{N}^{\#T_D}$, we introduce the following definition.

Definition 5.11 *We will say that* $\mathfrak{r} := (r_\alpha)_{\alpha\in T_D} \in \mathbb{N}^{\#T_D}$ *is an* admissible tuple for T_D, *if there exists* $\mathbf{v} \in \mathbf{V}_D \setminus \{\mathbf{0}\}$ *such that* $\dim U_\alpha^{\min}(\mathbf{v}) = r_\alpha$ *for all* $\alpha \in T_D \setminus \{D\}$.

Necessary conditions for $\mathfrak{r}$ to be admissible are

$$\begin{array}{ll} r_\alpha \le \dim V_j & \text{for } \alpha = \{j\} \in \mathcal{L}(T_D), \\ r_D = 1 & \text{for } \alpha = D. \end{array} \tag{5.6}$$

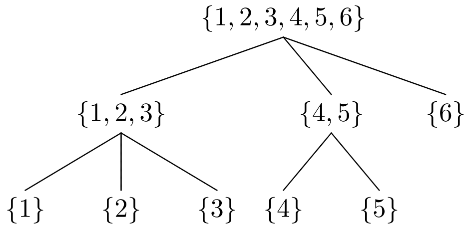

Figure 5.2: A dimension partition tree related with (5.5).

5.3 The representations of tensors of fixed TB rank

Let us consider for a given dimension partition tree T_D, a fixed admissible tuple $\mathfrak{r} \in \mathbb{N}^{\# T_D}$. Take $\mathbf{v} \in \mathbf{V}_D$ such that $\dim U_\alpha^{\min}(\mathbf{v}) = r_\alpha$ and consider a basis $\{\mathbf{u}_{i_\alpha}^{(\alpha)} : 1 \le i_\alpha \le r_\alpha\}$ of $U_\alpha^{\min}(\mathbf{v})$ for each $\alpha \in T_D \setminus \{D\}$. Since $\mathbf{v} \in {}_a\bigotimes_{\alpha \in S(D)} U_\alpha^{\min}(\mathbf{v})$, there exists $C^{(D)} \in \mathbb{R}_*^{\times_{\alpha\in S(D)} r_\alpha}$ such that

$$\mathbf{v} = \sum_{\substack{1\le i_\alpha \le r_\alpha\\ \alpha \in S(D)}} C^{(D)}_{(i_\alpha)_{\alpha\in S(D)}} \bigotimes_{\alpha \in S(D)} \mathbf{u}_{i_\alpha}^{(\alpha)}. \tag{5.7}$$

If $S(D) = \mathcal{L}(T_D)$, then (5.7) gives us the classical Tucker representation. Assume $S(D) \neq \mathcal{L}(T_D)$. Now, for each $\mu \in T_D \setminus \{D\}$ such that $S(\mu) \neq \emptyset$ we have $U_\mu^{\min}(\mathbf{v}) \subset {}_a\bigotimes_{\beta \in S(\mu)} U_\beta^{\min}(\mathbf{v})$ and then, there exists $C^{(\mu)} \in \mathbb{R}^{r_\mu \times \left(\times_{\beta\in S(\alpha)} r_\beta\right)}$ such that

$$\mathbf{u}_{i_\mu}^{(\mu)} = \sum_{\substack{1\le i_\beta \le r_\beta\\ \beta \in S(\alpha)}} C^{(\mu)}_{i_\mu,(i_\beta)_{\beta\in S(\mu)}} \bigotimes_{\beta \in S(\mu)} \mathbf{u}_{i_\beta}^{(\beta)}. \tag{5.8}$$

for $1 \le i_\mu \le r_\mu$. Since $\{\mathbf{u}_{i_\mu}^{(\mu)} : 1 \le i_\mu \le r_\mu\}$ is a basis, we can identify $C^{(\mu)}$ with a matrix, also denoted by $C^{(\mu)}$, in the non-compact Stiefel manifold $\mathbb{R}_*^{r_\mu \times \left(\prod_{\beta\in S(\mu)} r_\beta\right)}$, which is the set of matrices in $\mathbb{R}^{r_\mu \times \left(\prod_{\beta\in S(\alpha)} r_\beta\right)}$ whose rows are linearly independent (see 3.1.5 in [1]). From (5.7) and (5.8) we obtain the Tucker representation of $\mathbf{v}$, when $S(D) \neq \mathcal{L}(T_D)$, as

$$\mathbf{v} = \sum_{\substack{1\le i_k \le r_k\\ k \in \mathcal{L}(T_D)}} \left(\sum_{\substack{1\le i_\alpha \le r_\alpha\\ \alpha \in S(D)\\ \alpha \notin \mathcal{L}(T_D)}} C^{(D)}_{(i_\alpha)_{\alpha\in S(D)}} \prod_{\substack{\mu \in T_D\setminus\{D\}\\ S(\mu)\neq\emptyset}} C^{(\mu)}_{i_\mu,(i_\beta)_{\beta\in S(\mu)}} \right) \bigotimes_{k \in \mathcal{L}(T_D)} u_{i_k}^{(k)}. \tag{5.9}$$

The procedure, given a basis of $U_\alpha^{\min}(\mathbf{v})$ for $\alpha \in T_D \setminus \{D\}$, used to obtain (5.9) is completely characterised by a finite tuple of tensors

$$\mathfrak{C} := (C^{(\alpha)})_{\alpha \in T_D \setminus \mathcal{L}(T_D)} \in \bigtimes_{\alpha \in T_D\setminus\mathcal{L}(T_D)} \mathbb{R}^{r_\alpha \times \left(\times_{\beta\in S(\alpha)} r_\beta\right)},$$

where $C^{(D)} \in \mathbb{R}_*^{\times_{\alpha\in S(D)} r_\alpha}$ and $C^{(\mu)} \in \mathbb{R}_*^{r_\mu \times \left(\prod_{\beta\in S(\mu)} r_\beta\right)}$, for each $\mu \in T_D \setminus \{D\}$ such that $S(\mu) \neq \emptyset$. From now on, to simplify the notation, we introduce for an admissible $\mathfrak{r} \in \mathbb{N}^{T_D}$ the product vector space

$$\mathbb{R}^{\mathfrak{r}} := \bigtimes_{\alpha \in T_D\setminus\mathcal{L}(T_D)} \mathbb{R}^{r_\alpha \times \left(\times_{\beta\in S(\alpha)} r_\beta\right)}$$

and its open subset, and hence a manifold,

$$\mathbb{R}_*^{\mathfrak{r}} := \left\{ \mathfrak{C} \in \mathbb{R}^{\mathfrak{r}} : \begin{array}{l} C^{(D)} \in \mathbb{R}_*^{\times_{\alpha\in S(D)} r_\alpha} \text{and } C^{(\mu)} \in \mathbb{R}_*^{r_\mu \times \left(\prod_{\beta\in S(\mu)} r_\beta\right)} \\ \text{for each } \mu \in T_D \setminus \{D\} \text{ such that } S(\mu) \neq \emptyset. \end{array} \right\}.$$

Definition 5.12 *Let T_D be a given dimension partition tree and fix some tuple $\mathfrak{r} \in \mathbb{N}^{T_D}$ for T_D. The* set of TBF tensors of bounded TB rank $\mathfrak{r}$ *is defined by*

$$\mathcal{BT}_{\mathfrak{r}}(\mathbf{V}_D) := \left\{ \mathbf{v} \in \mathbf{V}_D : \ \dim \mathbf{U}_\alpha^{\min}(\mathbf{v}) \le r_\alpha \text{ for all } \alpha \in T_D \ \right\}, \tag{5.10}$$

and the set of TBF tensors of fixed TB rank $\mathfrak{r}$ *is defined by*

$$\mathcal{FT}_{\mathfrak{r}}(\mathbf{V}_D) := \left\{ \mathbf{v} \in \mathcal{BT}_{\mathfrak{r}}(\mathbf{V}_D) : \dim \mathbf{U}_\alpha^{\min}(\mathbf{v}) = r_\alpha \text{ for all } \alpha \in T_D \right\}. \tag{5.11}$$

Note that $\mathcal{FT}_{\mathfrak{r}}(\mathbf{V}_D) = \emptyset$ for an inadmissible tuple $\mathfrak{r}$. For $\mathfrak{r}, \mathfrak{s} \in \mathbb{N}^{T_D}$ we write $\mathfrak{s} \le \mathfrak{r}$ if and only if $s_\alpha \le r_\alpha$ for all $\alpha \in T_D$. Then we have

$$\mathcal{BT}_{\mathfrak{r}}(\mathbf{V}_D) = \bigcup_{\mathfrak{s} \le \mathfrak{r}} \mathcal{FT}_{\mathfrak{s}}(\mathbf{V}_D).$$

Next we give some useful examples.

Example 5.13 (Tucker format) *Consider the partition tree over $D := \{1, \ldots, d\}$, where $S(D) = \mathcal{L}(T_D) = \{\{j\} : 1 \le j \le d\}$. Let $(r_D, r_1, \ldots, r_d)$ be admissible, then $r_D = 1$ and $r_j \le \dim V_j$ for $1 \le j \le d$. Thus we can write*

$$\mathcal{BT}_{(1,r_1,\ldots,r_d)}(\mathbf{V}_D) = \mathcal{T}_{(r_1,\ldots,r_d)}(\mathbf{V}_D)$$

and

$$\mathcal{FT}_{(1,r_1,\ldots,r_d)}(\mathbf{V}_D) = \mathcal{M}_{(r_1,\ldots,r_d)}(\mathbf{V}_D).$$

Example 5.14 (Tensor Train format) *Consider a binary partition tree over $D := \{1, \ldots, d\}$ given by*

$$T_D = \{D, \{\{j\} : 1 \le j \le d\}, \{\{j+1, \ldots, d\} : 1 \le j \le d-2\}\}.$$

In particular, $S(\{j, \ldots, d\}) = \{\{j\}, \{j+1, \ldots, d\}\}$ for $1 \le j \le d-1$. This tensor based format is related to the following chain of inclusions:

$$\mathbf{U}_D^{\min}(\mathbf{v}) \subset \mathbf{U}_1^{\min}(\mathbf{v}) \otimes_a \mathbf{U}_{2\cdots d}^{\min}(\mathbf{v}) \subset \mathbf{U}_1^{\min}(\mathbf{v}) \otimes_a \mathbf{U}_2^{\min}(\mathbf{v}) \otimes_a \mathbf{U}_{3\cdots d}^{\min}(\mathbf{v}) \subset \cdots \subset {}_a\bigotimes_{j \in D} \mathbf{U}_j^{\min}(\mathbf{v}).$$

Finally, from Theorem 6.24 of [12], the following result can be shown.

Theorem 5.15 *Let $\mathbf{V}_{\|\cdot\|_D} = {}_{\|\cdot\|_D}\bigotimes_{j\in D} V_j$, be a tensor Banach space with a norm satisfying $\|\cdot\|_D \gtrsim \|\cdot\|_\vee$ and T_D be a dimension partition tree of the index set D. Then for each admissible tuple $\mathfrak{r} \in \mathbb{N}^{T_D}$ for T_D the following statements hold.*

(a) The set $\mathcal{BT}_{\mathfrak{r}}(\mathbf{V}_D)$ is weakly closed in $\mathbf{V}_{\|\cdot\|_D}$.

(b) Assume that $\mathbf{V}_{\|\cdot\|_D}$ is reflexive. Then for each $\mathbf{u} \in \mathbf{V}_{\|\cdot\|_D}$ there exists $\mathbf{v} \in \mathcal{BT}_{\mathfrak{r}}(\mathbf{V}_D)$ such that

$$\|\mathbf{u} - \mathbf{v}\| = \min_{\mathbf{w} \in \mathcal{BT}_{\mathfrak{r}}(\mathbf{V}_D)} \|\mathbf{u} - \mathbf{w}\|.$$

6 The manifold of TBF tensors of fixed TB rank

Now, assume that $\|\cdot\|_\alpha$ is a norm on $\mathbf{V}_\alpha = {}_a\bigotimes_{j\in\alpha} V_j$, for each $\alpha \in T_D \setminus \{D\}$, and $\mathbf{V}_{\alpha_{\|\cdot\|_\alpha}} = {}_{\|\cdot\|_\alpha}\bigotimes_{j\in\alpha} V_j$ is its corresponding tensor Banach space. Let

$$\mathbb{G}(T_D) := \bigtimes_{\alpha \in T_D \setminus \{D\}} \mathbb{G}(\mathbf{V}_{\alpha_{\|\cdot\|_\alpha}}) = \{\mathfrak{U} := \{U_\alpha\}_{\alpha \in T_D \setminus \{D\}} : U_\alpha \in \mathbb{G}(\mathbf{V}_{\|\cdot\|_\alpha})\}$$

be the product Banach manifold. Fix $\mathbf{v} \in \mathcal{FT}_{\mathfrak{r}}(\mathbf{V}_D)$ and consider a basis $\{\mathbf{u}_{i_\alpha}^{(\alpha)} : 1 \le i_\alpha \le r_\alpha\}$ of $U_\alpha^{\min}(\mathbf{v})$ for each $\alpha \in T_D \setminus \{D\}$ such that $\mathbf{v}$ can be represented by means (5.7) and (5.8). Thus $\mathbf{v}$ is completely characterised by $\mathfrak{C} \in \mathbb{R}_*^{\mathfrak{r}}$ and $(\{\mathbf{u}_{i_\alpha^{(\alpha)}} : 1 \le i_\alpha \le r_\alpha\})_{\alpha \in T_D \setminus \{D\}}$. Assume a decomposition into a direct sum

$$\mathbf{V}_{\alpha_{\|\cdot\|_\alpha}} = U_\alpha^{\min}(\mathbf{v}) \oplus W_\alpha^{\min}(\mathbf{v})$$

for $\alpha \in T_D \setminus \{D\}$. From Example 2.11 we recall for each $\alpha \in T_D \setminus \{D\}$ the existence of the set

$$\mathbb{G}(W_\alpha^{\min}(\mathbf{v}), \mathbf{V}_{\alpha_{\|\cdot\|_\alpha}}) = \{U_\alpha \in \mathbb{G}(\mathbf{V}_{\alpha_{\|\cdot\|_\alpha}}) : U_\alpha \oplus W_\alpha^{\min}(\mathbf{v}) = \mathbf{V}_{\alpha_{\|\cdot\|_\alpha}}\}$$

and the bijective map $\Psi_{U_\alpha^{\min}(\mathbf{v})\oplus W_\alpha^{\min}(\mathbf{v})} : \mathbb{G}(W_\alpha^{\min}(\mathbf{v}), \mathbf{V}_{\alpha_{\|\cdot\|_\alpha}}) \longrightarrow \mathcal{L}(U_\alpha^{\min}(\mathbf{v}), W_\alpha^{\min}(\mathbf{v}))$. Clearly, the map

$$\mathbf{\Psi}_\mathbf{v} : \bigtimes_{\alpha \in T_D \setminus \{D\}} \mathbb{G}(W_\alpha^{\min}(\mathbf{v}), \mathbf{V}_{\alpha_{\|\cdot\|_\alpha}}) \to \bigtimes_{\alpha \in T_D \setminus \{D\}} \mathcal{L}(U_\alpha^{\min}(\mathbf{v}), W_\alpha^{\min}(\mathbf{v})),$$

defined as $\mathbf{\Psi}_\mathbf{v} := \bigtimes_{\alpha \in T_D \setminus \{D\}} \Psi_{U_\alpha^{\min}(\mathbf{v})\oplus W_\alpha^{\min}(\mathbf{v})}$ is also bijective. Furthermore, it is a local chart for $\mathfrak{U}(\mathbf{v}) := \{U_\alpha^{\min}(\mathbf{v})\}_{\alpha \in T_D \setminus \{D\}}$ in $\mathbb{G}(T_D)$ such that $\mathbf{\Psi}_\mathbf{v}(\mathfrak{U}(\mathbf{v})) = \mathfrak{o} := (0)_{\alpha \in T_D \setminus \{D\}}$. To simplify the notation, for each $\mathbf{v} \in \mathcal{FT}_\mathfrak{r}(\mathbf{V}_D)$ we will use

$$\begin{aligned} \mathcal{L}_{T_D}(\mathbf{v}) &:= \bigtimes_{\alpha \in T_D \setminus \{D\}} \mathcal{L}\left(U_\alpha^{\min}(\mathbf{v}), W_\alpha^{\min}(\mathbf{v})\right) \\ &= \{\mathfrak{L} := \{L_\alpha\}_{\alpha \in T_D \setminus \{D\}} : L_\alpha \in \mathcal{L}\left(U_\alpha^{\min}(\mathbf{v}), W_\alpha^{\min}(\mathbf{v})\right)\}, \end{aligned}$$

which is a closed subspace of the Banach space

$$\mathcal{L}_{T_D} := \bigtimes_{\alpha \in T_D \setminus \{D\}} \mathcal{L}\left(\mathbf{V}_{\alpha_{\|\cdot\|_\alpha}}, \mathbf{V}_{\alpha_{\|\cdot\|_\alpha}}\right),$$

and

$$\mathbb{G}(\mathfrak{U}(\mathbf{v})) := \bigtimes_{\alpha \in T_D \setminus \{D\}} \mathbb{G}(W_\alpha^{\min}(\mathbf{v}), \mathbf{V}_{\alpha_{\|\cdot\|_\alpha}}),$$

which is a local neighbourhood of $\mathfrak{U}(\mathbf{v})$ in the manifold $\mathbb{G}(T_D)$. Moreover, $\mathfrak{U} = \mathbf{\Psi}_\mathbf{v}^{-1}(\mathfrak{L})$, with $U_\alpha = G(L_\alpha) = \left\{\mathbf{u}_\alpha + L_\alpha(\mathbf{u}_\alpha) : \mathbf{u}_\alpha \in U_\alpha^{\min}(\mathbf{v})\right\}$, for each $\alpha \in T_D \setminus \{D\}$. A useful result is the following.

Lemma 6.1 *For each $\alpha \in T_D \setminus \{D\}$, the set $\mathcal{L}(U_\alpha^{\min}(\mathbf{v}), W_\alpha^{\min}(\mathbf{v}))$ is a complemented subspace of $\mathcal{L}(\mathbf{V}_{\alpha_{\|\cdot\|_\alpha}}, \mathbf{V}_{\alpha_{\|\cdot\|_\alpha}})$, and hence for each $\mathbf{v} \in \mathbf{V}_D$, the set $\mathcal{L}_{T_D}(\mathbf{v})$ is a complemented subspace of $\mathcal{L}_{T_D}$.*

Proof. Observe that the map

$$\Pi_\alpha : \mathcal{L}\left(\mathbf{V}_{\alpha_{\|\cdot\|_\alpha}}, \mathbf{V}_{\alpha_{\|\cdot\|_\alpha}}\right) \to \mathcal{L}\left(\mathbf{V}_{\alpha_{\|\cdot\|_\alpha}}, \mathbf{V}_{\alpha_{\|\cdot\|_\alpha}}\right)$$

defined by

$$\Pi_\alpha(L_\alpha) = P_{W_\alpha^{\min}(\mathbf{v})\oplus U_\alpha^{\min}(\mathbf{v})} L_\alpha P_{U_\alpha^{\min}(\mathbf{v})\oplus W_\alpha^{\min}(\mathbf{v})}$$

is a projection onto $\mathcal{L}(U_\alpha^{\min}(\mathbf{v}), W_\alpha^{\min}(\mathbf{v}))$. ■

Now, we introduce the map

$$\Lambda_{T_D} : \mathcal{FT}_\mathfrak{r}(\mathbf{V}_D) \longrightarrow \mathbb{G}(T_D), \quad \mathbf{w} \mapsto \mathfrak{U}(\mathbf{w}) := (U_\alpha^{\min}(\mathbf{w}))_{\alpha \in T_D \setminus \{D\}},$$

and observe that for each $\mathbf{w} \in \mathcal{FT}_\mathfrak{r}(\mathbf{V}_D)$ we have

$$\Lambda_{T_D}^{-1}(\Lambda_{T_D}(\mathbf{w})) = \left\{\mathbf{u} \in \mathcal{FT}_\mathfrak{r}(\mathbf{V}_D) : U_\alpha^{\min}(\mathbf{u}) = U_\alpha^{\min}(\mathbf{w}) \text{ for all } \alpha \in T_D \setminus \{D\}\right\}.$$

We will define the local neighbourhood of $\mathbf{v}$, denoted by $\mathcal{U}(\mathbf{v})$, in $\mathcal{FT}_\mathfrak{r}(\mathbf{V}_D)$ as

$$\mathcal{U}(\mathbf{v}) := \Lambda_{T_D}^{-1}\left(\mathbb{G}(\mathfrak{U}(\mathbf{v}))\right) \subset \mathcal{FT}_\mathfrak{r}(\mathbf{V}_D).$$

Observe that for each $\mathbf{w} \in \mathcal{U}(\mathbf{v})$ we have

$$\mathbf{V}_{\alpha_{\|\cdot\|_\alpha}} = U_\alpha^{\min}(\mathbf{w}) \oplus W_\alpha^{\min}(\mathbf{v}),$$

where $U_\alpha^{\min}(\mathbf{w}) \in \mathbb{G}(W_\alpha^{\min}(\mathbf{v}), \mathbf{V}_{\|\cdot\|_\alpha})$, for each $\alpha \in T_D \setminus \{D\}$. Since

$$\mathbb{G}(W_\alpha^{\min}(\mathbf{v}), \mathbf{V}_{\alpha_{\|\cdot\|_\alpha}}) \overset{\Psi_{U_\alpha^{\min}(\mathbf{v})\oplus W_\alpha^{\min}(\mathbf{v})}}{\cong} \mathcal{L}(U_\alpha^{\min}(\mathbf{v}), W_\alpha^{\min}(\mathbf{v})),$$

there exists a unique $L_\alpha \in \mathcal{L}(U_\alpha^{\min}(\mathbf{v}), W_\alpha^{\min}(\mathbf{v}))$ such that

$$\Psi_{U_\alpha^{\min}(\mathbf{v})\oplus W_\alpha^{\min}(\mathbf{v})}(U_\alpha^{\min}(\mathbf{w})) = L_\alpha$$

for each $\alpha \in T_D \setminus \{D\}$. Moreover, we claim that

$$U_\alpha^{\min}(\mathbf{w}) = \operatorname{span}\{L_\alpha(\mathbf{u}_{i_\alpha}^{(\alpha)}) + \mathbf{u}_{i_\alpha}^{(\alpha)} : 1 \le i_\alpha \le r_\alpha\}$$

holds for all $\alpha \in T_D \setminus \{D\}$. To prove the claim, we only need to show that

$$\{L_\alpha(\mathbf{u}_{i_\alpha}^{(\alpha)}) + \mathbf{u}_{i_\alpha}^{(\alpha)} : 1 \le i_\alpha \le r_\alpha\}$$

are linearly independent in $U_\alpha^{\min}(\mathbf{w})$. If the last statement is not true, we may assume without loss of generality that

$$L_\alpha(\mathbf{u}_1^{(\alpha)}) + \mathbf{u}_1^{(\alpha)} = \sum_{k=2}^{r_\alpha} \lambda_k (L_\alpha(\mathbf{u}_k^{(\alpha)}) + \mathbf{u}_k^{(\alpha)}),$$

i.e.,

$$L_\alpha(\mathbf{u}_1^{(\alpha)}) - \sum_{k=2}^{r_\alpha} \lambda_k L_\alpha(\mathbf{u}_k^{(\alpha)}) = \sum_{k=2}^{r_\alpha} \lambda_k \mathbf{u}_k^{(\alpha)} - \mathbf{u}_1^{(\alpha)}.$$

The left-hand side is in $W_\alpha^{\min}(\mathbf{v})$ and the right-hand side is in $U_\alpha^{\min}(\mathbf{w})$. Since $W_\alpha^{\min}(\mathbf{v}) \cap U_\alpha^{\min}(\mathbf{w}) = \{\mathbf{0}\}$ we then have a contradiction and the claim follows.

For each $\mathbf{u} \in \Lambda_{T_D}^{-1}(\Lambda_{T_D}(\mathbf{w}))$ we fix the basis $\{\mathbf{w}_{i_\alpha}^{(\alpha)} := \mathbf{u}_{i_\alpha}^{(\alpha)} + L_\alpha(\mathbf{u}_{i_\alpha}^{(\alpha)}) : 1 \le i_\alpha \le r_\alpha\}$ of $U_\alpha^{\min}(\mathbf{w})$ for each $\alpha \in T_D \setminus \{D\}$. Then we define $\xi_{\mathbf{w}} : \Lambda_{T_D}^{-1}(\Lambda_{T_D}(\mathbf{w})) \longrightarrow \mathbb{R}_*^{\mathfrak{r}}$ by

$$\xi_{\mathbf{w}}(\mathbf{u}) := \mathfrak{C}(\mathbf{u}) = (C^{(\alpha)}(\mathbf{u}))_{\alpha \in T_D \setminus \mathcal{L}(T_D)},$$

where

$$\mathbf{u} = \sum_{\substack{1 \le i_\alpha \le r_\alpha \\ \alpha \in S(D)}} C^{(D)}_{(i_\alpha)_{\alpha \in S(D)}}(\mathbf{u}) \bigotimes_{\alpha \in S(D)} (L_\alpha(\mathbf{u}_{i_\alpha}^{(\alpha)}) + \mathbf{u}_{i_\alpha}^{(\alpha)})$$

and, if $S(D) \neq \mathcal{L}(T)$, for each $\mu \in T_D \setminus \{D\}$ such that $S(\mu) \neq \emptyset$ we have

$$L_\mu(\mathbf{u}_{i_\mu}^{(\mu)}) + \mathbf{u}_{i_\mu}^{(\mu)} = \sum_{\substack{1 \le i_\beta \le r_\beta \\ \beta \in S(\mu)}} C^{(\mu)}_{i_\mu,(i_\beta)_{\beta \in S(\mu)}}(\mathbf{u}) \bigotimes_{\beta \in S(\mu)} (L_\beta(\mathbf{u}_{i_\beta}^{(\beta)}) + \mathbf{u}_{i_\beta}^{(\beta)}),$$

for $1 \le i_\mu \le r_\mu$. Clearly, $\xi_{\mathbf{w}}$ is one-to-one. On the other hand, given $\mathfrak{B} \in \mathbb{R}_*^{\mathfrak{r}}$, we can construct $\mathbf{u} \in \Lambda_{T_D}^{-1}(\Lambda_{T_D}(\mathbf{w}))$ satisfying $\mathfrak{B} = \mathfrak{C}(\mathbf{u})$. Thus we can conclude that $\xi_{\mathbf{w}}$ is a bijection which is independent of $\mathbf{w}$.

It allows us to define a local chart $\Theta_{\mathbf{v}} : \mathcal{U}(\mathbf{v}) \longrightarrow \mathbb{R}_*^{\mathfrak{r}} \times \mathcal{L}_{T_D}(\mathbf{v})$ by

$$\Theta_{\mathbf{v}}(\mathbf{w}) := (\xi_{\mathbf{w}}(\mathbf{w}), \boldsymbol{\Psi}_{\mathbf{v}} \circ \Lambda_{T_D}(\mathbf{w})) = (\mathfrak{C}(\mathbf{w}), \boldsymbol{\Psi}_{\mathbf{v}}(\mathfrak{U}(\mathbf{w}))).$$

More precisely, $\Theta_{\mathbf{v}}(\mathbf{w}) = (\mathfrak{C}(\mathbf{w}), \mathfrak{L})$ if and only if

$$\mathbf{w} = \sum_{\substack{1 \le i_\alpha \le r_\alpha \\ \alpha \in S(D)}} C^{(D)}_{(i_\alpha)_{\alpha \in S(D)}}(\mathbf{w}) \bigotimes_{\alpha \in S(D)} (L_\alpha(\mathbf{u}_{i_\alpha}^{(\alpha)}) + \mathbf{u}_{i_\alpha}^{(\alpha)}), \tag{6.1}$$

where, if $S(D) \neq \mathcal{L}(T_D)$, for each $\mu \in T_D \setminus \{D\}$ such that $S(\mu) \neq \emptyset$ we have

$$L_\mu(\mathbf{u}_{i_\mu}^{(\mu)}) + \mathbf{u}_{i_\mu}^{(\mu)} = \sum_{\substack{1 \le i_\beta \le r_\beta \\ \beta \in S(\mu)}} C^{(\mu)}_{i_\mu,(i_\beta)_{\beta \in S(\mu)}}(\mathbf{w}) \bigotimes_{\beta \in S(\mu)} (L_\beta(\mathbf{u}_{i_\beta}^{(\beta)}) + \mathbf{u}_{i_\beta}^{(\beta)}) \tag{6.2}$$

for $1 \leq i_\mu \leq r_\mu$. Proceeding iteratively along the tree, we obtain, for $S(D) \neq \mathcal{L}(T_D)$, a Tucker format representation of $\mathbf{w}$ given by

$$\mathbf{w} = \sum_{\substack{1\leq i_k\leq r_k \\ k\in\mathcal{L}(T_D)}} \left(\sum_{\substack{1\leq i_\alpha\leq r_\alpha \\ \alpha\in S(D) \\ \alpha\notin\mathcal{L}(T_D)}} C^{(D)}_{(i_\alpha)_{\alpha\in S(D)}}(\mathbf{w}) \prod_{\substack{\mu\in T_D\setminus\{D\} \\ S(\mu)\neq\emptyset}} C^{(\mu)}_{i_\mu,(i_\beta)_{\beta\in S(\mu)}}(\mathbf{w}) \right) \bigotimes_{k\in\mathcal{L}(T_D)} (L_k(u^{(k)}_{i_k}) + u^{(k)}_{i_k}).$$

The next result shows that the collection $\{\Theta_{\mathbf{v}}, \mathcal{U}(\mathbf{v})\}_{\mathbf{v}\in\mathcal{FT}_{\mathfrak{r}}(\mathbf{V}_D)}$ is an atlas for $\mathcal{FT}_{\mathfrak{r}}(\mathbf{V}_D)$.

Theorem 6.2 *Assume that $\mathbf{V}_{\alpha_{\|\cdot\|_\alpha}}$ is a Banach space with norm $\|\cdot\|_\alpha$ for $\alpha \in T_D\setminus\{D\}$. Then the collection $\{\Theta_{\mathbf{v}}, \mathcal{U}(\mathbf{v})\}_{\mathbf{v}\in\mathcal{FT}_{\mathfrak{r}}(\mathbf{V}_D)}$ is an analytic atlas for $\mathcal{FT}_{\mathfrak{r}}(\mathbf{V}_D)$. Furthermore, the set $\mathcal{FT}_{\mathfrak{r}}(\mathbf{V}_D)$ of TBF tensors with fixed TB rank is an analytical Banach manifold.*

Proof. Clearly, $\{\mathcal{U}(\mathbf{v})\}_{\mathbf{v}\in\mathcal{FT}_{\mathfrak{r}}(\mathbf{V}_D)}$ is a covering of $\mathcal{FT}_{\mathfrak{r}}(\mathbf{V})$ and AT1 is true. Take $(\mathfrak{C},\mathfrak{L}) \in \mathbb{R}^{\mathfrak{r}}_* \times \mathcal{L}_{T_D}(\mathbf{v})$. By using (6.1)-(6.2), we can construct $\mathbf{w} \in \mathcal{U}(\mathbf{v})$ such that $\Theta_{\mathbf{v}}(\mathbf{w}) = (\mathfrak{C},\mathfrak{L})$, and in consequence $\Theta_{\mathbf{v}}$ is surjective. Now, consider that $\Theta_{\mathbf{v}}(\mathbf{u}) = \Theta_{\mathbf{v}}(\mathbf{w})$. Since $U^{\min}_\alpha(\mathbf{u}) = U^{\min}_\alpha(\mathbf{w})$ for all $\alpha \in T_D \setminus \{D\}$ and $\mathfrak{C}(\mathbf{v}) = \mathfrak{C}(\mathbf{w})$, also from (6.1)-(6.2) we can conclude that $\mathbf{w} = \mathbf{u}$. In consequence AT2 holds. Finally for $\mathbf{v},\mathbf{u} \in \mathcal{FT}_{\mathfrak{r}}(\mathbf{V}_D)$ consider $\mathcal{U}(\mathbf{v},\mathbf{u}) := \mathcal{U}(\mathbf{v}) \cap \mathcal{U}(\mathbf{u})$. Observe that $\mathbf{w} \in \mathcal{U}(\mathbf{v},\mathbf{u})$ if and only if

$$U^{\min}_\alpha(\mathbf{w}) \in \mathbb{G}(W^{\min}_\alpha(\mathbf{u}), \mathbf{V}_{\alpha_{\|\cdot\|_\alpha}}) \cap \mathbb{G}(W^{\min}_\alpha(\mathbf{v}), \mathbf{V}_{\alpha_{\|\cdot\|_\alpha}}) \text{ for } \alpha \in T_D.$$

Then we need to show that

$$\Theta_{\mathbf{v}} \circ \Theta^{-1}_{\mathbf{u}} : \Theta_{\mathbf{u}}(\mathcal{U}(\mathbf{v},\mathbf{u})) \longrightarrow \Theta_{\mathbf{v}}(\mathcal{U}(\mathbf{v},\mathbf{u}))$$

is a diffeomorphism. Take $(\mathfrak{C},\mathfrak{L}) \in \Theta_{\mathbf{u}}(\mathcal{U}(\mathbf{v},\mathbf{u}))$, such that $\Theta_{\mathbf{u}}(\mathbf{w}) = (\mathfrak{C},\mathfrak{L})$ for some $\mathbf{w} \in \mathcal{U}(\mathbf{v},\mathbf{u})$ and

$$\Theta_{\mathbf{v}} \circ \Theta^{-1}_{\mathbf{u}}(\mathfrak{C},\mathfrak{L}) = \Theta_{\mathbf{v}}(\mathbf{w}) = (\mathfrak{B},\mathfrak{N}).$$

Observe that

$$U^{\min}_\alpha(\mathbf{w}) = \operatorname{span}\{\mathbf{u}^{(\alpha)}_{i_\alpha} + L_\alpha(\mathbf{u}^{(\alpha)}_{i_\alpha}) : 1 \leq i_\alpha \leq r_\alpha\},$$

$$L_\alpha(\mathbf{u}^{(\alpha)}_{i_\alpha}) + \mathbf{u}^{(\alpha)}_{i_\alpha} = \sum_{\substack{1\leq i_\beta\leq r_\beta \\ \beta\in S(\alpha)}} C^{(\alpha)}_{i_\alpha,(i_\beta)_{\beta\in S(\mu)}}(\mathbf{w}) \bigotimes_{\beta\in S(\alpha)} (L_\beta(\mathbf{u}^{(\beta)}_{i_\beta}) + \mathbf{u}^{(\beta)}_{i_\beta}),$$

$$U^{\min}_\alpha(\mathbf{w}) = \operatorname{span}\{\mathbf{u}^{(\alpha)}_{i_\alpha} + N_\alpha(\mathbf{u}^{(\alpha)}_{i_\alpha}) : 1 \leq i_\alpha \leq r_\alpha\}$$

and

$$N_\alpha(\mathbf{u}^{(\alpha)}_{i_\alpha}) + \mathbf{u}^{(\alpha)}_{i_\alpha} = \sum_{\substack{1\leq i_\beta\leq r_\beta \\ \beta\in S(\alpha)}} B^{(\alpha)}_{i_\alpha,(i_\beta)_{\beta\in S(\mu)}}(\mathbf{w}) \bigotimes_{\beta\in S(\alpha)} (N_\beta(\mathbf{u}^{(\beta)}_{i_\beta}) + \mathbf{u}^{(\beta)}_{i_\beta})$$

holds for $1 \leq i_\alpha \leq r_\alpha$ and $\alpha \in T_D\setminus\{D\}$. Then it is possible to construct an isomorphism $S_\alpha : \mathbb{R}^{r_\alpha\times(\times_{\beta\in S(\alpha)} r_\beta)}_* \to \mathbb{R}^{r_\alpha\times(\times_{\beta\in S(\alpha)} r_\beta)}_*$ such that $S_\alpha(C^{(\alpha)}) = B^{(\alpha)}$ for each $\alpha \in T_D \setminus \{D\}$. Hence the map $\mathfrak{S} : \mathbb{R}^{\mathfrak{r}}_* \to \mathbb{R}^{\mathfrak{r}}_*$ given by $\mathfrak{S}(\mathfrak{C}) = \mathfrak{B}$ is also an isomorphism and we can write

$$\Theta_{\mathbf{v}} \circ \Theta^{-1}_{\mathbf{u}}(\mathfrak{C},\mathfrak{L}) = (\mathfrak{S}(\mathfrak{C}),\mathfrak{N}) = (\mathfrak{S}(\mathfrak{C}), \mathbf{\Psi}_{\mathbf{v}} \circ \Lambda_{T_D}(\mathbf{w}))).$$

Since $\Lambda_{T_D}(\mathbf{w}) = \mathfrak{U}(\mathbf{w})$ and $U^{\min}_\alpha(\mathbf{w}) = \Psi^{-1}_{U^{\min}_\alpha(\mathbf{u})\oplus W^{\min}_\alpha(\mathbf{u})}(L_\alpha)$ for each $\alpha \in T_D \setminus \{D\}$, we obtain

$$\Theta_{\mathbf{v}} \circ \Theta^{-1}_{\mathbf{u}}(\mathfrak{C},\mathfrak{L}) = \left(\mathfrak{S}(\mathfrak{C}), \left(\mathbf{\Psi}_{\mathbf{v}} \circ \mathbf{\Psi}^{-1}_{\mathbf{u}}\right)(\mathfrak{L})\right).$$

From [4] we know that $\Psi_{U^{\min}_\alpha(\mathbf{v})\oplus W^{\min}_\alpha(\mathbf{v})} \circ \Psi^{-1}_{U^{\min}_\alpha(\mathbf{u})\oplus W^{\min}_\alpha(\mathbf{u})}$ from

$$\Psi_{U^{\min}_\alpha(\mathbf{u})\oplus W^{\min}_\alpha(\mathbf{u})}\left(\mathbb{G}(W^{\min}_\alpha(\mathbf{u}), \mathbf{V}_{\alpha_{\|\cdot\|_\alpha}}) \cap \mathbb{G}(W^{\min}_\alpha(\mathbf{v}), \mathbf{V}_{\alpha_{\|\cdot\|_\alpha}})\right) \subset \mathcal{L}(U^{\min}_\alpha(\mathbf{u}), W^{\min}_\alpha(\mathbf{u}))$$

to

$$\Psi_{U_\alpha^{\min}(\mathbf{v})\oplus W_\alpha^{\min}(\mathbf{v})}\left(\mathbb{G}(W_\alpha^{\min}(\mathbf{u}),\mathbf{V}_{\alpha_{\|\cdot\|_\alpha}})\cap\mathbb{G}(W_\alpha^{\min}(\mathbf{v}),\mathbf{V}_{\alpha_{\|\cdot\|_\alpha}})\right)\subset\mathcal{L}(U_\alpha^{\min}(\mathbf{v}),W_\alpha^{\min}(\mathbf{v}))$$

is an analytic diffeomorphism for each $\alpha\in T_D\setminus\{D\}$. Then $\mathbf{\Psi_v}\circ\mathbf{\Psi_u}^{-1}$ is an analytic diffeomorphism from

$$\mathbf{\Psi_u}\left(\bigtimes_{\alpha\in T_D\setminus\{D\}}\left(\mathbb{G}(W_\alpha^{\min}(\mathbf{u}),\mathbf{V}_{\alpha_{\|\cdot\|_\alpha}})\cap\mathbb{G}(W_\alpha^{\min}(\mathbf{v}),\mathbf{V}_{\alpha_{\|\cdot\|_\alpha}})\right)\right)\subset\mathcal{L}_{T_D}(\mathbf{u})$$

to

$$\mathbf{\Psi_v}\left(\bigtimes_{\alpha\in T_D\setminus\{D\}}\left(\mathbb{G}(W_\alpha^{\min}(\mathbf{u}),\mathbf{V}_{\alpha_{\|\cdot\|_\alpha}})\cap\mathbb{G}(W_\alpha^{\min}(\mathbf{v}),\mathbf{V}_{\alpha_{\|\cdot\|_\alpha}}\right)\right)\subset\mathcal{L}_{T_D}(\mathbf{v}).$$

Clearly, AT3 holds and the theorem follows. ∎

By using the geometric structure of local charts for the manifold $\mathcal{FT}_{\mathfrak{r}}(\mathbf{v})$, we can identify its tangent space at $\mathbf{v}$ with $\mathbb{T}_{\mathbf{v}}(\mathcal{FT}_{\mathfrak{r}}(\mathbf{V}_D)):=\mathbb{R}^{\mathfrak{r}}\times\mathcal{L}_{T_D}(\mathbf{v})$. We will consider $\mathbb{T}_{\mathbf{v}}(\mathcal{FT}_{\mathfrak{r}}(\mathbf{V}_D))$ endowed with the product norm

$$|||(\mathfrak{C},\mathfrak{L})|||:=\sum_{\alpha\in T_D\setminus\mathcal{L}(T_D)}\|C^{(\alpha)}\|_F+\sum_{\alpha\in T_D\setminus\{D\}}\|L_\alpha\|_{W_\alpha^{\min}(\mathbf{v})\leftarrow U_\alpha^{\min}(\mathbf{v})}.$$

with $\|\cdot\|_F$ the Frobenius norm.

Note that $\mathcal{L}(\mathbf{V}_{\alpha_{\|\cdot\|_\alpha}},\mathbf{V}_{\alpha_{\|\cdot\|_\alpha}})$ endowed with the norm $\|\cdot\|_{\mathbf{V}_{\alpha_{\|\cdot\|_\alpha}}\leftarrow\mathbf{V}_{\alpha_{\|\cdot\|_\alpha}}}$ is a Banach space. Thus, even if $\mathbf{V}_{\|\cdot\|_\alpha}$ is a Hilbert space for all $\alpha\in T_D\setminus\{D\}$, the set $\mathcal{L}_{T_D}$ is a Banach space. The following lemma allows us to identify $L_\alpha\in\mathcal{L}\left(U_\alpha^{\min}(\mathbf{v}),W_\alpha^{\min}(\mathbf{v})\right)$ with a vector in $W_\alpha^{\min}(\mathbf{v})^{\dim U_\alpha^{\min}(\mathbf{v})}$ for each $\alpha\in T_D\setminus\{D\}$.

Lemma 6.3 *Assume that* $\mathbf{V}_{\alpha_{\|\cdot\|_\alpha}}={}_{\|\cdot\|_\alpha}\bigotimes_{j\in\alpha}V_j$ *is a Banach space for* $\alpha\in T_D\setminus\{D\}$*. Then for each* $\mathbf{v}\in\mathbf{V}_\alpha={}_a\bigotimes_{j\in\alpha}V_j$ *the Banach space* $\mathcal{L}(U_\alpha^{\min}(\mathbf{v}),W_\alpha^{\min}(\mathbf{v}))$ *is linearly isomorphic to* $W_\alpha^{\min}(\mathbf{v})^{\dim U_\alpha^{\min}(\mathbf{v})}$.

Proof. Since $\mathbf{v}\in\mathbf{V}_\alpha$, then $\dim U_\alpha^{\min}(\mathbf{v})=r_\alpha<\infty$ and every $L_\alpha\in\mathcal{L}\left(U_\alpha^{\min}(\mathbf{v}),W_\alpha^{\min}(\mathbf{v})\right)$ is a finite rank operator, i.e., $\dim L_\alpha(U_\alpha^{\min}(\mathbf{v}))<\infty$. In consequence, $\mathcal{L}\left(U_\alpha^{\min}(\mathbf{v}),W_\alpha^{\min}(\mathbf{v})\right)$ is linearly isomorphic to $U_\alpha^{\min}(\mathbf{v})^*\otimes_a W_\alpha^{\min}(\mathbf{v})$ by means the canonical isomorphism

$$\Xi_\alpha\left(\sum_{k=1}^n\varphi_k^{(\alpha)}\otimes w_k^{(\alpha)}\right)(u^{(\alpha)}):=\sum_{k=1}^n\varphi_k^{(\alpha)}(u^{(\alpha)})w_k^{(\alpha)},$$

(see, e.g., Proposition 16.8 in [5]). Moreover, $U_\alpha^{\min}(\mathbf{v})^*\otimes_a W_\alpha^{\min}(\mathbf{v})$ is linearly isomorphic to $\mathbb{R}^{r_\alpha}\otimes_a W_\alpha^{\min}(\mathbf{v})=W_\alpha^{\min}(\mathbf{v})^{r_\alpha}$ (simply consider $(x_k)_{k=1}^{r_\alpha}\otimes w=(x_kw)_{k=1}^{r_\alpha}$) and the lemma follows. ∎

Corollary 6.4 *Assume that* $\mathbf{V}_{\alpha_{\|\cdot\|_\alpha}}$ *is a Hilbert space with norm* $\|\cdot\|_\alpha$ *for* $\alpha\in T_D\setminus\{D\}$*. Then* $\mathcal{FT}_{\mathfrak{r}}(\mathbf{V}_D)$ *is an analytical Hilbert manifold.*

Proof. Lemma 6.3 allows us to identify each $L_\alpha\in\mathcal{L}\left(U_\alpha^{\min}(\mathbf{v}),W_\alpha^{\min}(\mathbf{v})\right)$ with a $(\mathbf{w}_{s_\alpha}^{(\alpha)})_{s_\alpha=1}^{s_\alpha=r_\alpha}\in W_\alpha^{\min}(\mathbf{v})^{r_\alpha}$, where $\mathbf{w}_{s_\alpha}^{(\alpha)}=L_\alpha(\mathbf{u}_{(s_\alpha)}^\alpha)$ and $U_\alpha^{\min}(\mathbf{v})=\operatorname{span}\{\mathbf{u}_{(1)}^\alpha,\ldots,\mathbf{u}_{(r_\alpha)}^\alpha\}$, for $\alpha\in T_D\setminus\{D\}$. Thus we can identify each $(\mathfrak{C},\mathfrak{L})\in\mathcal{U}(\mathbf{v})$ with a pair

$$(\mathfrak{C},\mathfrak{W})\in\mathbb{R}_*^{\mathfrak{r}}\times\bigtimes_{\alpha\in T_D\setminus\{D\}}W_\alpha^{\min}(\mathbf{v})^{r_\alpha},$$

where $\mathfrak{W}:=((\mathbf{w}_{s_\alpha}^{(\alpha)})_{s_\alpha=1}^{s_\alpha=r_\alpha})_{\alpha\in T_D\setminus\{D\}}$. We assume that $\mathbb{R}_*^{\mathfrak{r}}\times\bigtimes_{\alpha\in T_D\setminus\{D\}}W_\alpha^{\min}(\mathbf{v})^{r_\alpha}$ is an open subset of the Hilbert space $\mathbb{R}^{\mathfrak{r}}\times\bigtimes_{\alpha\in T_D\setminus\{D\}}W_\alpha^{\min}(\mathbf{v})^{r_\alpha}$ endowed with the product norm

$$\|(\mathfrak{C},\mathfrak{W})\|_\times:=\sum_{\alpha\in T_D\setminus\mathcal{L}(T_D)}\|C^\alpha\|_F+\sum_{\alpha\in T_D\setminus\{D\}}\sum_{s_\alpha=1}^{r_\alpha}\|\mathbf{w}_{s_\alpha}^{(\alpha)}\|_\alpha.$$

It allows us to define local charts, also denoted by $\Theta_{\mathbf{v}}$, by

$$\Theta_{\mathbf{v}}^{-1} : \mathbb{R}_*^{\mathfrak{r}} \times \bigtimes_{\alpha \in T_D \setminus \{D\}} W_\alpha^{\min}(\mathbf{v})^{r_\alpha} \longrightarrow \mathcal{U}(\mathbf{v}),$$

where $\Theta_{\mathbf{v}}^{-1}(\mathfrak{C}, \mathfrak{W}) = \mathbf{w}$. Here $\mathbf{w}$ is given by (6.1)–(6.2) putting $L_\alpha(\mathbf{u}_{i_\alpha}^{(\alpha)}) = \mathbf{w}_{i_\alpha}^{(\alpha)}$, $1 \le i_\alpha \le r_\alpha$ and $\alpha \in T_D \setminus \{D\}$. Since each local chart is defined over an open subset of the Hilbert space $\mathbb{R}^{\mathfrak{r}} \times \bigtimes_{\alpha \in T_D \setminus \{D\}} W_\alpha^{\min}(\mathbf{v})^{r_\alpha}$, the corollary follows. ■

7 $\mathcal{FT}_{\mathfrak{r}}(\mathbf{V}_D)$ as an embedded submanifold

Let $\mathbf{V}_{\|\cdot\|_D} := \overline{\mathbf{V}_D}^{\|\cdot\|_D}$ be a tensor Banach space, where $\|\cdot\|_D$ is a norm, and consider in $\mathcal{FT}_{\mathfrak{r}}(\mathbf{V}_D)$ the topology induced by the norm $\|\cdot\|_D$. The natural ambient space for $\mathcal{FT}_{\mathfrak{r}}(\mathbf{V}_D)$ is the Banach tensor space $\mathbf{V}_{\|\cdot\|_D}$. Since the natural inclusion $i : \mathcal{FT}_{\mathfrak{r}}(\mathbf{V}_D) \longrightarrow \mathbf{V}_{\|\cdot\|_D}$, given by $i(\mathbf{v}) = \mathbf{v}$, is a homeomorphism onto its image, we will study i as a map between Banach manifolds. To this end we introduce the following definitions.

Definition 7.1 *Let X and Y two Banach manifolds. Let $F : X \to Y$ be a map. We shall say that F is a $\mathcal{C}^r$* morphism *if given $x \in X$ there exists a chart (U, φ) at x and a chart (W, ψ) at $F(x)$ such that $F(U) \subset W$, and the map*

$$\psi \circ F \circ \varphi^{-1} : \varphi(U) \to \psi(W)$$

is a $\mathcal{C}^r$-Fréchet differentiable map.

To describe i as a morphism, we proceed as follows. Given $\mathbf{v} \in \mathcal{FT}_{\mathfrak{r}}(\mathbf{V}_D)$, we consider $\mathcal{U}(\mathbf{v})$, a local neighbourhood of $\mathbf{v}$, and then

$$i \circ \Theta_{\mathbf{v}}^{-1} : \mathbb{R}_*^{\mathfrak{r}} \times \mathcal{L}_{T_D}(\mathbf{v}) \to \mathbf{V}_{\|\cdot\|_D}, \quad (\mathfrak{C}, \mathfrak{L}) \mapsto \sum_{\substack{1 \le i_\alpha \le r_\alpha \\ \alpha \in S(D)}} C^{(D)}_{(i_\alpha)_{\alpha \in S(D)}} \bigotimes_{\alpha \in S(D)} (L_\alpha(\mathbf{u}_{i_\alpha}^{(\alpha)}) + \mathbf{u}_{i_\alpha}^{(\alpha)}),$$

where for each $\mu \in T_D \setminus \{D\}$ such that $S(\mu) \neq \emptyset$ we have

$$L_\mu(\mathbf{u}_{i_\mu}^{(\mu)}) + \mathbf{u}_{i_\mu}^{(\mu)} = \sum_{\substack{1 \le i_\beta \le r_\beta \\ \beta \in S(\mu)}} C^{(\mu)}_{i_\mu, (i_\beta)_{\beta \in S(\mu)}} \bigotimes_{\beta \in S(\mu)} (L_\beta(\mathbf{u}_{i_\beta}^{(\beta)}) + \mathbf{u}_{i_\beta}^{(\beta)})$$

for $1 \le i_\mu \le r_\mu$.

Our next step is to recall the definition of the differential as a morphism which gives a linear map between the tangent spaces of the manifolds involved with the morphism.

Definition 7.2 *Let X and Y two Banach manifolds. Let $F : X \to Y$ be a $\mathcal{C}^r$ morphism, i.e.,*

$$\psi \circ F \circ \varphi^{-1} : \varphi(U) \to \psi(W)$$

is a $\mathcal{C}^r$-Fréchet differentiable map, where (U, φ) is a chart in X at x and (W, ψ) is a chart in Y at $F(x)$. For $x \in X$, we define

$$\mathrm{T}_x F : \mathbb{T}_x(X) \longrightarrow \mathbb{T}_{F(x)}(Y), \quad v \mapsto [(\psi \circ F \circ \varphi^{-1})'(\varphi(x))]v.$$

Assume that $i \circ \Theta_{\mathbf{v}}^{-1}$ is Fréchet differentiable, then $\mathrm{T}_{\mathbf{v}} i : \mathbb{R}^{\mathfrak{r}} \times \mathcal{L}_{T_D}(\mathbf{v}) \to \mathbf{V}_{\|\cdot\|_D}$, is given by

$$\mathrm{T}_{\mathbf{v}} i(\dot{\mathfrak{C}}, \dot{\mathfrak{L}}) = [(i \circ \Theta_{\mathbf{v}}^{-1})'(\Theta_{\mathbf{v}}(\mathbf{v}))](\dot{\mathfrak{C}}, \dot{\mathfrak{L}}).$$

The next lemma describes the tangent map $\mathrm{T}_{\mathbf{v}} i$.

Lemma 7.3 *Let* $\mathbf{v} \in \mathcal{FT}_{\mathfrak{r}}(\mathbf{V}_D)$ *be such that* $\Theta_{\mathbf{v}}(\mathbf{v}) = (\mathfrak{C}(\mathbf{v}), \mathfrak{o})$*, where* $\mathfrak{C}(\mathbf{v}) = (C^{(\alpha)})_{\alpha \in T_D \setminus \mathcal{L}(T_D)}$*,* $\mathfrak{o} = (0)_{\alpha \in T_D \setminus \{D\}}$ *and*

$$U_\alpha^{\min}(\mathbf{v}) = \operatorname{span}\{\mathbf{u}_{i_\alpha}^{(\alpha)} : 1 \le i_\alpha \le r_\alpha\}$$

for $\alpha \in T_D \setminus \{D\}$*. Then the following statements hold.*

(a) Assume that the tensor product map $\bigotimes$ *is continuous. Then the map* $i \circ \Theta_{\mathbf{v}}^{-1}$ *from* $\mathbb{R}_*^{\mathfrak{r}} \times \mathcal{L}_{T_D}(\mathbf{v})$ *to* $\mathbf{V}_{\|\cdot\|_D}$ *is Fréchet differentiable, and hence* $\mathrm{T}_{\mathbf{v}} i \in \mathcal{L}\left(\mathbb{T}_{\mathbf{v}}(\mathcal{FT}_{\mathfrak{r}}(\mathbf{V}_D)), \mathbf{V}_{\|\cdot\|_D}\right)$*.*

(b) Assume $(\dot{\mathfrak{C}}, \dot{\mathfrak{L}}) \in \mathbb{T}_{\mathbf{v}}(\mathcal{FT}_{\mathfrak{r}}(\mathbf{V}_D))$*, where* $\dot{\mathfrak{C}} = (\dot{C}^{(\alpha)})_{\alpha \in T_D \setminus \mathcal{L}(T_D)}$ *and* $\dot{\mathfrak{L}} = (\dot{L}_\alpha)_{\alpha \in T_D \setminus \{D\}}$*. Then* $\mathbf{w} = \mathrm{T}_{\mathbf{v}} i(\dot{\mathfrak{C}}, \dot{\mathfrak{L}})$ *if and only*

$$\mathbf{w} = \sum_{\substack{1 \le i_\alpha \le r_\alpha \\ \alpha \in S(D)}} \dot{C}^{(D)}_{(i_\alpha)_{\alpha \in S(D)}} \bigotimes_{\alpha \in S(D)} \mathbf{u}_{i_\alpha}^{(\alpha)} + \sum_{\mu \in S(D)} \sum_{\substack{1 \le i_\alpha \le r_\alpha \\ \alpha \in S(D)}} C^{(D)}_{(i_\alpha)_{\alpha \in S(D)}} \left(\dot{L}_\mu(\mathbf{u}_{i_\mu}^{(\mu)}) \otimes \bigotimes_{\substack{\alpha \in S(D) \\ \alpha \ne \mu}} \mathbf{u}_{i_\alpha}^{(\alpha)} \right),$$

where for each $\gamma \in T_D \setminus \{D\}$ *with* $S(\gamma) \neq \emptyset$*,*

$$\dot{L}_\gamma(\mathbf{u}_{i_\gamma}^{(\gamma)}) = \sum_{\substack{1 \le i_\beta \le r_\beta \\ \beta \in S(\gamma)}} \dot{C}^{(\gamma)}_{i_\gamma, (i_\beta)_{\beta \in S(\gamma)}} \bigotimes_{\beta \in S(\gamma)} \mathbf{u}_{i_\beta}^{(\beta)} + \sum_{\delta \in S(\gamma)} \sum_{\substack{1 \le i_\beta \le r_\beta \\ \beta \in S(\gamma)}} C^{(\gamma)}_{i_\gamma, (i_\beta)_{\beta \in S(\gamma)}} \left(\dot{L}_\delta(\mathbf{u}_{i_\delta}^{(\delta)}) \otimes \bigotimes_{\substack{\beta \ne \delta \\ \beta \in S(\gamma)}} \mathbf{u}_{i_\beta}^{(\beta)} \right)$$

holds for $1 \le i_\gamma \le r_\gamma$*.*

Proof. To prove statement (a), observe that for each $\mathbf{u}_\alpha \in U_\alpha^{\min}(\mathbf{v})$, $\alpha \in T_D \setminus \{D\}$, the map

$$\Phi_{\mathbf{u}_\alpha} : \mathcal{L}(U_\alpha^{\min}(\mathbf{v}), W_\alpha^{\min}(\mathbf{v})) \to W_\alpha^{\min}(\mathbf{v}), \quad L_\alpha \mapsto L_\alpha(\mathbf{u}_\alpha),$$

is linear and continuous, and hence Fréchet differentiable. Clearly, its differential is given by $[\Phi'_{\mathbf{u}_\alpha}(L_\alpha)](H_\alpha) = H_\alpha(\mathbf{u}_\alpha)$. Thus, if the tensor product map $\bigotimes$ is continuous, by Proposition 3.5 it is also Fréchet differentiable. Then, by the chain rule, the map $\Theta_{\mathbf{v}}^{-1}$ is also Fréchet differentiable. Since $\mathrm{T}_{\mathbf{v}} i(\dot{\mathfrak{C}}, \dot{\mathfrak{L}}) = [(i \circ \Theta_{\mathbf{v}}^{-1})'(\mathfrak{C}, \mathfrak{o})](\dot{\mathfrak{C}}, \dot{\mathfrak{L}})$, (a) follows. Statement (b) follows by using the chain rule. ■

Next we recall the definition of an immersion between manifolds.

Definition 7.4 *Let* $F : X \to Y$ *be a morphism between Banach manifolds and let* $x \in X$*. We shall say that* F *is an* immersion *at* x*, if there exists an open neighbourhood* X_1 *of* x *in* X *such that the restriction of* F *to* X_1 *induces an isomorphism of* X_1 *onto a submanifold of* Y*. We say that* F *is an* immersion *if it is an immersion at each point of* X*.*

For manifolds modelled on Banach spaces we have the following criterion for immersions (see Proposition 2.2 in [16]).

Proposition 7.5 *Let* X, Y *be Banach manifolds of class* $\mathcal{C}^p$ $(p \ge 1)$*. Let* $F : X \to Y$ *be a* $\mathcal{C}^p$ *morphism and* $x \in X$*. Then* F *is an immersion at* x *if and only if* $\mathrm{T}_x F$ *is injective and* $\mathrm{T}_x F(\mathbb{T}_x(X))$ *is a complemented subspace.*

A related concept with an immersion between Banach manifolds is the following. Assume that X and Y are Banach manifolds and let $f : X \longrightarrow Y$ be a $\mathcal{C}^r$ morphism. If f is an injective immersion, then $f(X)$ is called an *immersed manifold of* Y.

Recall that there exists injective immersions which are not isomorphisms onto manifolds. It allows us to introduce the following definition.

Definition 7.6 *An injective immersion* $f : X \longrightarrow Y$*, i.e., a homeomorphism onto* $f(X)$ *with the relative topology induced from* Y *is called an* embedding*. Moreover, if* $f : X \longrightarrow Y$ *is an embedding, then* $f(X)$ *is called an* embedded submanifold*.*

From Lemma 7.3(b) we known that $\mathrm{T}_{\mathbf{v}}i$ is injective. Thus, to show that i is an inmersion, and hence an embedding, we need to prove that $\mathbb{T}_{\mathbf{v}}(\mathcal{FT}_{\mathfrak{r}}(\mathbf{V}_D))$ is a complemented subspace of $\mathbf{V}_{\|\cdot\|_D}$. In the next result, that will be useful, we prove that, if $\|\cdot\|_D$ is a uniform crossnorm, then $\mathrm{T}_{\mathbf{v}}i$ is a linear isomorphism from $\mathbb{T}_{\mathbf{v}}(\mathcal{FT}_{\mathfrak{r}}(\mathbf{V}_D))$ to a linear closed subspace of $\mathbf{V}_{\|\cdot\|_D}$. It allows us to identify the tangent space $\mathbb{T}_{\mathbf{v}}(\mathcal{FT}_{\mathfrak{r}}(\mathbf{V}_D))$ with a closed subspace of $\mathbf{V}_{\|\cdot\|_D}$.

Theorem 7.7 *Let $\mathbf{V}_{\|\cdot\|_D}$ be a tensor Banach space such that $\|\cdot\|_D$ is a uniform crossnorm. Then for each $\mathbf{v}\in\mathcal{FT}_{\mathfrak{r}}(\mathbf{V}_D)$, the set $\mathrm{T}_{\mathbf{v}}i\,(\mathbb{T}_{\mathbf{v}}(\mathcal{FT}_{\mathfrak{r}}(\mathbf{V}_D)))$ is a closed subspace of $\mathbf{V}_{\|\cdot\|_D}$ linearly isomorphic to the Banach space $\mathbb{T}_{\mathbf{v}}(\mathcal{FT}_{\mathfrak{r}}(\mathbf{V}_D))$.*

Proof. To prove that $\mathrm{T}_{\mathbf{v}}i\,(\mathbb{T}_{\mathbf{v}}(\mathcal{FT}_{\mathfrak{r}}))$ is a closed subspace in $\mathbf{V}_{\|\cdot\|}$, take a sequence

$$(\dot{\mathfrak{C}}_n,\dot{\mathfrak{L}}_n):=\left((\dot{C}_n^{(\alpha)})_{\alpha\in T_D\setminus\mathcal{L}(T_D)},(\dot{L}_\alpha^{(n)})_{\alpha\in T_D\setminus\{D\}}\right),$$

in $\mathbb{T}_{\mathbf{v}}(\mathcal{FT}_{\mathfrak{r}}(\mathbf{V}_D))$ such that $\mathbf{w}_n:=\mathrm{T}_{\mathbf{v}}i((\dot{\mathfrak{C}}_n,\dot{\mathfrak{L}}_n))\longrightarrow\mathbf{w}$ as $n\to\infty$. We want to show the existence of a $(\dot{\mathfrak{C}},\dot{\mathfrak{L}})\in\mathbb{T}_{\mathbf{v}}(\mathcal{FT}_{\mathfrak{r}}\mathbf{V}_D)$ such that $\mathbf{w}=\mathrm{T}_{\mathbf{v}}i((\dot{\mathfrak{C}},\dot{\mathfrak{L}}))$. To this end, from Lemma 7.3, we may assume

$$\mathbf{w}_n=\sum_{\substack{1\le i_\alpha\le r_\alpha\\ \alpha\in S(D)}}(\dot{C}_n^{(D)})_{(i_\alpha)_{\alpha\in S(D)}}\bigotimes_{\alpha\in S(D)}\mathbf{u}_{i_\alpha}^{(\alpha)}+\sum_{\mu\in S(D)}\sum_{\substack{1\le i_\alpha\le r_\alpha\\ \alpha\in S(D)}}(C^{(D)})_{(i_\alpha)_{\alpha\in S(D)}}\left(\dot{L}_\mu^{(n)}(\mathbf{u}_{i_\mu}^{(\mu)})\otimes\bigotimes_{\substack{\alpha\in S(D)\\ \alpha\neq\mu}}\mathbf{u}_{i_\alpha}^{(\alpha)}\right),$$

where for each $\gamma\in T_D\setminus\{D\}$ such that $S(\gamma)\neq 0$, we have

$$\dot{L}_\gamma^{(n)}(\mathbf{u}_{i_\gamma}^{(\gamma)})=\sum_{\substack{1\le i_\beta\le r_\beta\\ \beta\in S(\gamma)}}(\dot{C}_n^{(\gamma)})_{i_\gamma,(i_\beta)_{\beta\in S(\gamma)}}\bigotimes_{\beta\in S(\mu)}\mathbf{u}_{i_\beta}^{(\beta)}+\sum_{\delta\in S(\gamma)}\sum_{\substack{1\le i_\beta\le r_\beta\\ \beta\in S(\gamma)}}C^{(\gamma)}_{i_\gamma,(i_\beta)_{\beta\in S(\gamma)}}\left(\dot{L}_\delta^{(n)}(\mathbf{u}_{i_\delta}^{(\delta)})\otimes\bigotimes_{\substack{\beta\neq\delta\\ \beta\in S(\gamma)}}\mathbf{u}_{i_\beta}^{(\beta)}\right),$$

for $1\le i_\gamma\le r_\gamma$.

To prove the theorem we will show that for each $\gamma\in T_D$ such that $S(\gamma)\neq 0$, there exist $\dot{C}^{(\gamma)}\in\mathbb{R}^{\times_{\alpha\in S(\gamma)}r_\alpha}$ and $(\dot{L}_\mu)_{\mu\in S(\gamma)}\in\times_{\alpha\in S(\gamma)}\mathcal{L}(U_\alpha^{\min}(\mathbf{v}),W_\alpha^{\min}(\mathbf{v}))$ such that $\dot{C}_n^{(\gamma)}\to\dot{C}^{(\gamma)}$ and $(\dot{L}_\mu^{(n)})_{\mu\in S(\gamma)}\to(\dot{L}_\mu)_{\mu\in S(\gamma)}$ as $n\to\infty$. In consequence, taking

$$(\dot{\mathfrak{C}},\dot{\mathfrak{L}}):=\left((\dot{C}^{(\alpha)})_{\alpha\in T_D\setminus\mathcal{L}(T_D)},(\dot{L}_\alpha)_{\alpha\in T_D\setminus\{D\}}\right),$$

we have $(\dot{\mathfrak{C}}_n,\dot{\mathfrak{L}}_n)\to(\dot{\mathfrak{C}},\dot{\mathfrak{L}})$ as $n\to\infty$ and the continuity of $\mathrm{T}_{\mathbf{v}}i$ proves the theorem.

To show this, we proceed inductively along the tree. First, assume $\gamma=D$. To simplify the notation, we denote[6]

$$\mathbf{z}(\dot{C}_n^{(D)}):=\sum_{\substack{1\le i_\alpha\le r_\alpha\\ \alpha\in S(D)}}(\dot{C}_n^{(D)})_{(i_\alpha)_{\alpha\in S(D)}}\bigotimes_{\alpha\in S(D)}\mathbf{u}_{i_\alpha}^{(\alpha)}$$

and

$$\mathbf{z}(\dot{L}_\mu^{(n)}):=\sum_{\substack{1\le i_\alpha\le r_\alpha\\ \alpha\in S(D)}}(C_n^{(D)})_{(i_\alpha)_{\alpha\in S(D)}}\left(\dot{L}_\mu^{(n)}(\mathbf{u}_{i_\mu}^{(\mu)})\otimes\bigotimes_{\substack{\alpha\in S(D)\\ \alpha\neq\mu}}\mathbf{u}_{i_\alpha}^{(\alpha)}\right)$$

for each $\mu\in S(D)$, such that

$$\mathbf{w}_n=\mathbf{z}(\dot{C}_n^{(D)})+\sum_{\mu\in S(D)}\mathbf{z}(\dot{L}_\mu^{(n)}).$$

[6] We separate the case D from other nodes (see later), since the notations are different.

We introduce the following linear and bounded maps

$$\mathcal{P}^{(S(D))} = \overline{\bigotimes_{\alpha\in S(D)} P_{U_\alpha^{\min}(\mathbf{v})\oplus W_\alpha^{\min}(\mathbf{v})}}, \quad \mathcal{P}_\mu^{(S(D))} := \overline{P_{W_\mu^{\min}(\mathbf{v})\oplus U_\mu^{\min}(\mathbf{v})} \otimes \mathcal{P}_{[\mu]}^{(S(D))}},$$

where

$$\mathcal{P}_{[\mu]}^{(S(D))} := \bigotimes_{\beta\in S(D),\beta\neq\mu} P_{U_\beta^{\min}(\mathbf{v})\oplus W_\beta^{\min}(\mathbf{v})}.$$

Clearly, $\mathcal{P}^{(S(D))}(\mathbf{w}_n) = \mathbf{z}(\dot{C}_n^{(D)})$ and $\mathcal{P}_\mu^{(S(D))}(\mathbf{w}_n) = \mathbf{z}(\dot{L}_\mu^{(n)})$, for each $\mu \in S(D)$. Hence

$$\left(\mathcal{P}^{(S(D))} + \sum_{\mu\in S(D)} \mathcal{P}_\mu^{(S(D))}\right)(\mathbf{w}_n) = \mathbf{w}_n.$$

Since $\mathbf{w}_n \to \mathbf{w}$ we have

$$\mathbf{z}(\dot{C}_n^{(D)}) = \sum_{\substack{1\le i_\alpha\le r_\alpha\\ \alpha\in S(D)}} (\dot{C}_n^{(D)})_{(i_\alpha)_{\alpha\in S(D)}} \bigotimes_{\alpha\in S(D)} \mathbf{u}_{i_\alpha}^{(\alpha)} \to \mathcal{P}^{(S(D))}(\mathbf{w}).$$

Then there exists $\dot{C}^{(D)} \in \mathbb{R}^{\times_{\alpha\in S(D)} r_\alpha}$ such that

$$\mathcal{P}^{(S(D))}(\mathbf{w}) = \sum_{\substack{1\le i_\alpha\le r_\alpha\\ \alpha\in S(D)}} (\dot{C}^{(D)})_{(i_\alpha)_{\alpha\in S(D)}} \bigotimes_{\alpha\in S(D)} \mathbf{u}_{i_\alpha}^{(\alpha)},$$

and $\dot{C}_n^{(D)} \to \dot{C}^{(D)}$ as $n \to \infty$.

On the other hand,

$$\mathbf{z}(\dot{L}_\mu^{(n)}) = \sum_{\substack{1\le i_\alpha\le r_\alpha\\ \alpha\in S(D)}} C^{(D)}_{(i_\alpha)_{\alpha\in S(D)}} \left(\dot{L}_\mu^{(n)}(\mathbf{u}_{i_\mu}^{(\mu)}) \otimes \bigotimes_{\substack{\alpha\in S(D)\\ \alpha\neq\mu}} \mathbf{u}_{i_\alpha}^{(\alpha)}\right) \to \mathcal{P}_\mu^{(S(D))}(\mathbf{w}),$$

for each $\mu \in S(D)$. We claim, for each fixed $\mu \in S(D)$, the existence of

$$\{w_{i_\mu} : 1 \le i_\mu \le r_\mu\} \subset W_\mu^{\min}(\mathbf{v})$$

such that

$$\mathcal{P}_\mu^{(S(D))}(\mathbf{w}) = \sum_{\substack{1\le i_\beta\le r_\beta\\ \beta\in S(D)}} C^{(D)}_{(i_\beta)_{\beta\in S(D)}} w_{i_\mu} \otimes \bigotimes_{\substack{\beta\in S(D)\\ \beta\neq\mu}} \mathbf{u}_{i_\beta}^{(\beta)}.$$

To prove the claim, observe that

$$\mathbf{z}(\dot{L}_\mu^{(n)}) \in \mathcal{T}_{(r_\alpha)_{\alpha\in S(D)}}\left(W_\mu^{\min}(\mathbf{v}) \otimes_a \left({}_a\bigotimes_{\substack{\beta\neq\mu\\ \beta\in S(D)}} U_\beta^{\min}(\mathbf{v})\right)\right),$$

which is a weakly closed set in $\mathbf{V}_{\|\cdot\|_D}$, and hence closed. In consequence, there exist

$$\{w'_{i_\mu} : 1 \le i_\mu \le r_\mu\} \subset W_\mu^{\min}(\mathbf{v})$$

and $S^{(D)} \in \mathbb{R}^{\times_{\alpha\in S(D)} r_\alpha}$ such that

$$\mathcal{P}_\mu^{(S(D))}(\mathbf{w}) = \sum_{\substack{1\le i_\beta\le r_\beta\\ \beta\in S(D)}} S^{(D)}_{(i_\beta)_{\beta\in S(D)}} w'_{i_\mu} \otimes \bigotimes_{\substack{\beta\in S(D)\\ \beta\neq\mu}} \mathbf{u}_{i_\beta}^{(\beta)}.$$

Fix $1 \leq i'_\mu \leq r_\mu$ and let $(i_\alpha)_{\alpha\in S(D)}$ be such that $i_\mu = i'_\mu$. Now, take $\boldsymbol{\varphi}_{[i'_\mu]} \in {}_a\bigotimes_{\substack{\alpha\in S(D)\\ \alpha\neq\mu}} V_\alpha^*$ such that $\boldsymbol{\varphi}_{[i'_\mu]}(\bigotimes_{\substack{\alpha\in S(D)\\ \alpha\neq\mu}} \mathbf{u}_{i_\alpha}^{(\alpha)}) = 1$ and $\boldsymbol{\varphi}_{[i'_\mu]}(\bigotimes_{\substack{\alpha\in S(D)\\ \alpha\neq\mu}} \mathbf{u}_{i'_\alpha}^{(\alpha)}) = 0$ for all $(i'_\alpha)_{\alpha\in S(D)} \neq (i_\alpha)_{\alpha\in S(D)}$. Since

$$\mathbf{z}(\dot{L}_\mu^{(n)}) - \mathcal{P}_\mu^{(S(D))}(\mathbf{w}) \to \mathbf{0},$$

then $(id_\mu \otimes \boldsymbol{\varphi}_{[i'_\mu]})\left(\mathbf{z}(\dot{L}_\mu^{(n)}) - \mathcal{P}_\mu^{(S(D))}(\mathbf{w})\right) \to 0 \in V_\mu$. Thus, $C^{(D)}_{(i_\alpha)_{\alpha\in S(D)}} \dot{L}_\mu^{(n)}(\mathbf{u}_{i_\mu}) \to S^{(D)}_{(i_\alpha)_{\alpha\in S(D)}} w'_{i_\mu}$. It is clear that if $C^{(D)}_{(i_\alpha)_{\alpha\in S(D)}} = 0$ then $S^{(D)}_{(i_\alpha)_{\alpha\in S(D)}} = 0$. Otherwise,

$$\dot{L}_\mu^{(n)}(\mathbf{u}_{i_\mu}^{(\mu)}) \to \frac{S^{(D)}_{(i_\alpha)_{\alpha\in S(D)}}}{C^{(D)}_{(i_\alpha)_{\alpha\in S(D)}}} w'_{i_\mu},$$

when $C^{(D)}_{(i_\alpha)_{\alpha\in S(D)}} \neq 0$. Introduce

$$w_{i_\mu} := \begin{cases} \frac{S^{(D)}_{(i_\alpha)_{\alpha\in S(D)}}}{C^{(D)}_{(i_\alpha)_{\alpha\in S(D)}}} w'_{i_\mu} & \text{if } C^{(D)}_{(i_\alpha)_{\alpha\in S(D)}} \neq 0 \\ 0 & \text{otherwise,} \end{cases}$$

and the claim follows. Finally, define the linear map $\dot{L}_\mu \in \mathcal{L}(U_\mu^{\min}(\mathbf{v}), W_\mu^{\min}(\mathbf{v}))$ by $\dot{L}_\mu(\mathbf{u}_{i_\mu}) := w_{i_\mu}$ for each $1 \leq i_\mu \leq r_\mu$, and hence

$$\mathcal{P}_\mu^{(S(D))}(\mathbf{w}) = \sum_{\substack{1\leq i_\beta\leq r_\beta\\ \beta\in S(D)}} C^{(D)}_{(i_\beta)_{\beta\in S(D)}} \left(\dot{L}_\mu(\mathbf{u}_{i_\mu}) \otimes \bigotimes_{\substack{\beta\in S(D)\\ \beta\neq\mu}} \mathbf{u}_{i_\beta}^{(\beta)} \right).$$

Moreover, $\dot{L}_\mu^{(n)} \to \dot{L}_\mu$ as $n \to \infty$.

Let $\mu \in T_D \setminus \{D\}$ such that $S(\mu) \neq \emptyset$. Then $\dot{C}_n^{(\mu)} \to \dot{C}^{(\mu)}$ and $\dot{L}_\gamma^{(n)} \to \dot{L}_\gamma$ for all $\gamma \in S(\mu)$ as $n \to \infty$. In particular, we have

$$\dot{L}_\gamma^{(n)}(\mathbf{u}_{i_\gamma}^{(\gamma)}) \to \dot{L}_\gamma(\mathbf{u}_{i_\gamma}^{(\gamma)})$$

for $1 \leq i_\gamma \leq r_\gamma$. Observe that for each $\gamma \in S(\mu)$ such that $S(\gamma) \neq \emptyset$ we write

$$\dot{L}_\gamma^{(n)}(\mathbf{u}_{i_\gamma}^{(\gamma)}) = \mathbf{z}_{i_\gamma}(\dot{C}_n^{(\gamma)}) + \sum_{\delta\in S(\gamma)} \mathbf{z}_{i_\gamma}(L_\delta^{(n)}),$$

where

$$\mathbf{z}_{i_\gamma}(\dot{C}_n^{(\gamma)}) := \sum_{\substack{1\leq i_\alpha\leq r_\alpha\\ \alpha\in S(\gamma)}} (\dot{C}_n^{(\gamma)})_{(i_\alpha)_{\alpha\in S(\gamma)}} \bigotimes_{\alpha\in S(\gamma)} \mathbf{u}_{i_\alpha}^{(\alpha)}$$

and

$$\mathbf{z}_{i_\gamma}(\dot{L}_\mu^{(n)}) := \sum_{\substack{1\leq i_\alpha\leq r_\alpha\\ \alpha\in S(\gamma)}} (C_n^{(\gamma)})_{i_\gamma,(i_\alpha)_{\alpha\in S(\gamma)}} \left(\dot{L}_\mu^{(n)}(\mathbf{u}_{i_\mu}^{(\mu)}) \otimes \bigotimes_{\substack{\alpha\in S(\gamma)\\ \alpha\neq\mu}} \mathbf{u}_{i_\alpha}^{(\alpha)} \right)$$

for each $\mu \in S(\gamma)$. Thus we can repeat the proof substituting D by γ, obtaining $\dot{C}^\gamma$ and $(\dot{L}_\delta)_{\delta\in S(\gamma)}$ such that $\dot{C}_n^\gamma \to \dot{C}^\gamma$, $\dot{L}_\delta^{(n)} \to \dot{L}_\delta$, where for each $\delta \in S(\gamma)$ such that $S(\delta) \neq \emptyset$ we have

$$\dot{L}_\delta(\mathbf{u}_{i_\delta}^{(\delta)}) = \sum_{\substack{1\leq i_\beta\leq r_\beta\\ \beta\in S(\delta)}} \dot{C}^{(\gamma)}_{(i_\beta)_{\beta\in S(\gamma)}} \bigotimes_{\beta\in S(\gamma)} \mathbf{u}_{i_\beta}^{(\beta)} + \sum_{\eta\in S(\gamma)} \sum_{\substack{1\leq i_\beta\leq r_\beta\\ \beta\in S(\gamma)}} C^{(\gamma)}_{(i_\beta)_{\beta\in S(\gamma)}} \left(\dot{L}_\eta(\mathbf{u}_{i_\eta}^{(\eta)}) \otimes \bigotimes_{\substack{\beta\neq\eta\\ \beta\in S(\gamma)}} \mathbf{u}_{i_\beta}^{(\beta)} \right)$$

for $1 \le i_\delta \le r_\delta$. This ends the proof of the theorem. ■

Since in a Hilbert space every linear subspace is closed if and only if it is complemented, by Proposition 7.5 and Theorem 7.7, we obtain the following result.

Corollary 7.8 *Let $\mathbf{V}_{\|\cdot\|_D}$ be a tensor Hilbert space such that $\|\cdot\|_D$ is a uniform crossnorm. Then $\mathcal{FT}_{\mathfrak{r}}(\mathbf{V}_D)$ is an embedded submanifold of $\mathbf{V}_{\|\cdot\|_D}$. Moreover, the canonical scalar product (8.9) is uniform.*

Finally, we also have the next corollary.

Corollary 7.9 *Let $\mathbf{V}_{\|\cdot\|_D}$ be a reflexive tensor Banach space such that $\|\cdot\|_D$ is a uniform crossnorm and take $\mathbf{v} \in \mathcal{FT}_{\mathfrak{r}}(\mathbf{V}_D)$. Then for each $\dot{\mathbf{u}} \in \mathbf{V}_{\|\cdot\|_D}$ there exists $\dot{\mathbf{v}}_{best} \in \mathrm{T}_{\mathbf{v}} i\left(\mathbb{T}_{\mathbf{v}}(\mathcal{FT}_{\mathfrak{r}}(\mathbf{V}_D))\right)$ such that*

$$\|\dot{\mathbf{u}} - \dot{\mathbf{v}}_{best}\| = \min_{\dot{\mathbf{v}} \in \mathrm{T}_{\mathbf{v}} i(\mathbb{T}_{\mathbf{v}}(\mathcal{FT}_{\mathfrak{r}}(\mathbf{V}_D)))} \|\dot{\mathbf{u}} - \dot{\mathbf{v}}\|. \tag{7.1}$$

8 On the Dirac–Frenkel variational principle on tensor Banach spaces

8.1 Model Reduction in tensor Banach spaces

In this section we consider the abstract ordinary differential equation in a reflexive tensor Banach space $\mathbf{V}_{\|\cdot\|_D}$, with a uniform crossnorm $\|\cdot\|_D$, given by

$$\dot{\mathbf{u}}(t) = \mathbf{F}(t, \mathbf{u}(t)), \text{ for } t \ge 0 \tag{8.1}$$

$$\mathbf{u}(0) = \mathbf{u}_0, \tag{8.2}$$

where we assume $\mathbf{u}_0 \neq \mathbf{0}$ and $\mathbf{F} : [0, \infty) \times \mathbf{V}_{\|\cdot\|_D} \longrightarrow \mathbf{V}_{\|\cdot\|_D}$ satisfying the usual conditions. We want (or: 'would like') to approach $\mathbf{u}(t)$, for $t \in I := (0, \varepsilon)$ for some $\varepsilon > 0$, by a differentiable curve $t \mapsto \mathbf{v}_r(t)$ from I to $\mathcal{FT}_{\mathbf{r}}(\mathbf{V}_D)$, where $\mathfrak{r} \in \mathbb{N}^{T_D}$ is such that $\mathbf{v}_r(0) = \mathbf{v}_0$ satisfies

$$\|\mathbf{u}_0 - \mathbf{v}_0\|_D = \min_{\mathbf{w} \in \mathcal{FT}_{\mathfrak{r}}(\mathbf{V}_D)} \|\mathbf{u}_0 - \mathbf{w}\|_D = \min_{\mathbf{w} \in \mathcal{BT}_{\mathfrak{r}}(\mathbf{V}_D)} \|\mathbf{u}_0 - \mathbf{w}\|_D.$$

Our main goal is to construct a Reduced Order Model of (8.1)–(8.2) over the Banach manifold $\mathcal{FT}_{\mathfrak{r}}(\mathbf{V}_D)$. Since $\mathbf{F}(t, \mathbf{v}_r(t))$ in $\mathbf{V}_{\|\cdot\|_D}$, for each $t \in I$, and $\mathrm{T}_{\mathbf{v}_r(t)} i\left(\mathbb{T}_{\mathbf{v}_r(t)}(\mathcal{FT}_{\mathfrak{r}}(\mathbf{V}_D))\right)$ is a closed linear subspace in $\mathbf{V}_{\|\cdot\|_D}$, we have the existence of a $\dot{\mathbf{v}}_r(t) \in \mathrm{T}_{\mathbf{v}_r(t)} i\left(\mathbb{T}_{\mathbf{v}_r(t)}(\mathcal{FT}_{\mathfrak{r}}(\mathbf{V}_D))\right)$ such that

$$\|\dot{\mathbf{v}}_r(t) - \mathbf{F}(t, \mathbf{v}_r(t))\|_D = \min_{\dot{\mathbf{v}}(t) \in \mathrm{T}_{\mathbf{v}_r(t)} i\left(\mathbb{T}_{\mathbf{v}_r(t)}(\mathcal{FT}_{\mathfrak{r}}(\mathbf{V}_D))\right)} \|\dot{\mathbf{v}}(t) - \mathbf{F}(t, \mathbf{v}_r(t))\|_D,$$

It is well-known that, if $\mathbf{V}_{\|\cdot\|_D}$ is a Hilbert space, then $\dot{\mathbf{v}}_r(t) = \mathcal{P}_{\mathbf{v}_r(t)}(\mathbf{F}(t, \mathbf{v}_r(t)))$, where

$$\mathcal{P}_{\mathbf{v}_r(t)} = \mathcal{P}_{\mathrm{T}_{\mathbf{v}_r(t)} i\left(\mathbb{T}_{\mathbf{v}_r(t)}(\mathcal{FT}_{\mathfrak{r}}(\mathbf{V}_D))\right) \oplus \mathrm{T}_{\mathbf{v}_r(t)} i\left(\mathbb{T}_{\mathbf{v}_r(t)}(\mathcal{FT}_{\mathfrak{r}}(\mathbf{V}_D))\right)^{\perp}}$$

is called the *metric projection.* It has the following important property: $\dot{\mathbf{v}}_r(t) = \mathcal{P}_{\mathbf{v}_r(t)}(\mathbf{F}(t, \mathbf{v}_r(t)))$ if and only if

$$\langle \dot{\mathbf{v}}_r(t) - \mathbf{F}(t, \mathbf{v}_r(t)), \dot{\mathbf{v}}(t) \rangle_D = 0 \text{ for all } \dot{\mathbf{v}}(t) \in \mathrm{T}_{\mathbf{v}_r(t)} i\left(\mathbb{T}_{\mathbf{v}_r(t)}(\mathcal{FT}_{\mathfrak{r}}(\mathbf{V}_D))\right).$$

The concept of a metric projection can be extended to the Banach setting. To this end we recall the following definitions. A Banach space X with norm $\|\cdot\|$ is said to be *strictly convex* if $\|x+y\|/2 < 1$ for all $x, y \in X$ with $\|x\| = \|y\| = 1$ and $x \neq y$. It is *uniformly convex* if $\lim_{n\to\infty} \|x_n - y_n\| = 0$ for any two sequences $\{x_n\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$ such that $\|x_n\| = \|y_n\| = 1$ and $\lim_{n\to\infty} \|x_n + y_n\|/2 = 1$. It is known that a uniformly convex Banach space is reflexive and strictly convex. A Banach space X is said to be *smooth* if the limit

$$\lim_{t\to 0} \frac{\|x + ty\| - \|x\|}{t}$$

exists for all $x, y \in U := \{z \in X : \|z\| = 1\}$. Finally, a Banach space X is said to be *uniformly smooth* if its modulus of smoothness

$$\rho(\tau) = \sup_{x,y \in U} \left\{ \frac{\|x + \tau y\| + \|x - \tau y\|}{2} - 1 \right\}, \ \tau > 0,$$

satisfies the condition $\lim_{\tau \to 0} \rho(\tau) = 0$. In uniformly smooth spaces, and only in such spaces, the norm is uniformly Fréchet differentiable. A uniformly smooth Banach space is smooth. The converse is true if the Banach space is finite-dimensional. It is known that the space L^p $(1 < p < \infty)$ is a uniformly convex and uniformly smooth Banach space.

Let $\langle \cdot, \cdot \rangle : X \times X^* \longrightarrow \mathbb{R}$ denote the duality map, i.e.,

$$\langle x, f \rangle := f(x).$$

The normalised duality mapping $J : X \longrightarrow 2^{X^*}$ is defined by

$$J(x) := \{f \in X^* : \langle x, f \rangle = \|x\|^2 = (\|f\|^*)^2\}.$$

Notice that, in a Hilbert space, the duality mapping is the identity operator. The duality mapping J has the following properties (see [2]):

(a) If X is smooth, the map J is single-valued;

(b) if X is smooth, then J is norm–to–weak* continuous;

(c) if X is uniformly smooth, then J is uniformly norm–to–norm continuous on each bounded subset of X.

Assume that $\mathbf{V}_{\|\cdot\|_D}$ is a reflexive and strictly convex tensor Banach space with a uniform crossnorm $\|\cdot\|_D$. For $\mathbf{F}(t, \mathbf{v}_r(t))$ in $\mathbf{V}_{\|\cdot\|_D}$, with a fixed $t \in I$, it is known that the set

$$\left\{ \dot{\mathbf{v}}_r(t) : \|\dot{\mathbf{v}}_r(t) - \mathbf{F}(t, \mathbf{v}_r(t))\|_D = \min_{\dot{\mathbf{v}}(t) \in \mathrm{T}_{\mathbf{v}_r(t)} i\left(\mathbb{T}_{\mathbf{v}_r(t)}(\mathcal{FT}_{\mathfrak{r}}(\mathbf{V}_D))\right)} \|\dot{\mathbf{v}}(t) - \mathbf{F}(t, \mathbf{v}_r(t))\|_D \right\}$$

is always a singleton. Let $\mathcal{P}_{\mathbf{v}_r(t)}$ be the mapping of $\mathbf{V}_{\|\cdot\|_D}$ onto $\mathrm{T}_{\mathbf{v}_r(t)} i\left(\mathbb{T}_{\mathbf{v}_r(t)}(\mathcal{FT}_{\mathfrak{r}}(\mathbf{V}_D))\right)$ defined by $\dot{\mathbf{v}}_r(t) := \mathcal{P}_{\mathbf{v}_r(t)}(\mathbf{F}(t, \mathbf{v}_r(t)))$ if and only if

$$\|\dot{\mathbf{v}}_r(t) - \mathbf{F}(t, \mathbf{v}_r(t))\|_D = \min_{\dot{\mathbf{v}}(t) \in \mathrm{T}_{\mathbf{v}_r(t)} i\left(\mathbb{T}_{\mathbf{v}_r(t)}(\mathcal{FT}_{\mathfrak{r}}(\mathbf{V}_D))\right)} \|\dot{\mathbf{v}}(t) - \mathbf{F}(t, \mathbf{v}_r(t))\|_D.$$

It is also called *the metric projection.* The classical characterisation of the metric projection allows us to state the next result.

Theorem 8.1 *Assume that $\mathbf{V}_{\|\cdot\|_D}$ is a reflexive and strictly convex tensor Banach space with a uniform crossnorm $\|\cdot\|_D$. Then for each $t \in I$ we have*

$$\dot{\mathbf{v}}_r(t) = \mathcal{P}_{\mathbf{v}_r(t)}(\mathbf{F}(t, \mathbf{v}_r(t))) \tag{8.3}$$

if and only if

$$\langle \dot{\mathbf{v}}_r(t) - \dot{\mathbf{v}}(t), J(\mathbf{F}(t, \mathbf{v}_r(t)) - \dot{\mathbf{v}}_r(t)) \rangle \geq 0 \text{ for all } \dot{\mathbf{v}}(t) \in \mathrm{T}_{\mathbf{v}_r(t)} i\left(\mathbb{T}_{\mathbf{v}_r(t)}(\mathcal{FT}_{\mathfrak{r}}(\mathbf{V}_D))\right). \tag{8.4}$$

An alternative approach is the use of the so-called *generalised projection operator* (see also [2]). To formulate this, we will use the following framework. Assume now that $\mathbf{V}_{\|\cdot\|_D}$ is a reflexive, strictly convex and smooth tensor Banach space with a uniform crossnorm $\|\cdot\|_D$. Following [14], we can define a function $\phi : \mathbf{V}_{\|\cdot\|_D} \times \mathbf{V}_{\|\cdot\|_D} \longrightarrow \mathbb{R}$ by

$$\phi(\mathbf{u}, \mathbf{v}) = \|\mathbf{u}\|_D^2 - 2\langle \mathbf{u}, J(\mathbf{v}) \rangle + \|\mathbf{v}\|_D^2,$$

where $\langle\cdot,\cdot\rangle$ denotes the duality map and J is the normalised duality mapping. It is known that the set

$$\left\{\dot{\mathbf{v}}_r(t) : \phi(\dot{\mathbf{v}}_r(t), \mathbf{F}(t,\mathbf{v}_r(t))) = \min_{\dot{\mathbf{v}}(t)\in \mathrm{T}_{\mathbf{v}_r(t)}i\left(\mathbb{T}_{\mathbf{v}_r(t)}(\mathcal{FT}_{\mathfrak{r}}(\mathbf{V}_D))\right)} \phi(\dot{\mathbf{v}}(t), \mathbf{F}(t,\mathbf{v}_r(t)))\right\}$$

is always a singleton. It allows us to define a map $\Pi_{\mathbf{v}_r(t)} : \mathbf{V}_{\|\cdot\|_D} \longrightarrow \mathrm{T}_{\mathbf{v}_r(t)}i\left(\mathbb{T}_{\mathbf{v}_r(t)}(\mathcal{FT}_{\mathfrak{r}}(\mathbf{V}_D))\right)$ by $\dot{\mathbf{v}}_r(t) := \Pi_{\mathbf{v}_r(t)}(\mathbf{F}(t,\mathbf{v}_r(t)))$ if and only if

$$\phi(\dot{\mathbf{v}}_r(t), \mathbf{F}(t,\mathbf{v}_r(t))) = \min_{\dot{\mathbf{v}}(t)\in \mathrm{T}_{\mathbf{v}_r(t)}i\left(\mathbb{T}_{\mathbf{v}_r(t)}(\mathcal{FT}_{\mathfrak{r}}(\mathbf{V}_D))\right)} \phi(\dot{\mathbf{v}}(t), \mathbf{F}(t,\mathbf{v}_r(t))).$$

The map $\Pi_{\mathbf{v}_r(t)}$ is called *the generalised projection.*

Remark 8.2 *Emphasize also that, in general, the operators $\mathcal{P}_{\mathbf{v}_r(t)}$ and $\Pi_{\mathbf{v}_r(t)}$ are nonlinear in Banach (not Hilbert) spaces.*

Again, a classical characterisation of the generalised projection give us the following theorem.

Theorem 8.3 *Assume that $\mathbf{V}_{\|\cdot\|_D}$ is a reflexive, strictly convex and smooth tensor Banach space with a uniform crossnorm $\|\cdot\|_D$. Then for each $t\in I$ we have*

$$\dot{\mathbf{v}}_r(t) = \Pi_{\mathbf{v}_r(t)}(\mathbf{F}(t,\mathbf{v}_r(t))) \tag{8.5}$$

if and only if

$$\langle \dot{\mathbf{v}}_r(t) - \dot{\mathbf{v}}(t), J(\mathbf{F}(t,\mathbf{v}_r(t))) - J(\dot{\mathbf{v}}_r(t))\rangle \geq 0 \text{ for all } \dot{\mathbf{v}}(t) \in \mathrm{T}_{\mathbf{v}_r(t)}i\left(\mathbb{T}_{\mathbf{v}_r(t)}(\mathcal{FT}_{\mathfrak{r}}(\mathbf{V}_D))\right). \tag{8.6}$$

The next corollary is a consequence either of Theorem 8.1 or Theorem 8.3.

Corollary 8.4 *Assume that $\mathbf{V}_{\|\cdot\|_D}$ is a tensor Hilbert space with a uniform crossnorm $\|\cdot\|_D$. Then for each $t\in I$ we have*

$$\dot{\mathbf{v}}_r(t) = \mathcal{P}_{\mathbf{v}_r(t)}(\mathbf{F}(t,\mathbf{v}_r(t))) \tag{8.7}$$

if and only if

$$\langle \dot{\mathbf{v}}_r(t) - \mathbf{F}(t,\mathbf{v}_r(t)), \dot{\mathbf{v}}(t)\rangle_D = 0 \text{ for all } \dot{\mathbf{v}}(t) \in \mathrm{T}_{\mathbf{v}(t)}i\left(\mathbb{T}_{\mathbf{v}(t)}(\mathcal{FT}_{\mathfrak{r}}(\mathbf{V}_D))\right). \tag{8.8}$$

8.2 The time–dependent Hartree method

Let $\langle\cdot,\cdot\rangle_j$ be a scalar product defined on V_j ($1\le j\le d$), i.e., V_j is a pre-Hilbert space. Then $\mathbf{V} = {}_a\bigotimes_{j=1}^d V_j$ is again a pre-Hilbert space with a scalar product which is defined for elementary tensors $\mathbf{v} = \bigotimes_{j=1}^d v^{(j)}$ and $\mathbf{w} = \bigotimes_{j=1}^d w^{(j)}$ by

$$\langle \mathbf{v},\mathbf{w}\rangle = \left\langle \bigotimes_{j=1}^d v^{(j)}, \bigotimes_{j=1}^d w^{(j)}\right\rangle := \prod_{j=1}^d \left\langle v^{(j)}, w^{(j)}\right\rangle_j \qquad \text{for all } v^{(j)}, w^{(j)} \in V_j. \tag{8.9}$$

This bilinear form has a unique extension $\langle\cdot,\cdot\rangle : \mathbf{V}\times\mathbf{V}\to\mathbb{R}$. One verifies that $\langle\cdot,\cdot\rangle$ is a scalar product, called the *induced scalar product.* Let $\mathbf{V}$ be equipped with the norm $\|\cdot\|$ corresponding to the induced scalar product $\langle\cdot,\cdot\rangle$. As usual, the Hilbert tensor space $\mathbf{V}_{\|\cdot\|} = {}_{\|\cdot\|}\bigotimes_{j=1}^d V_j$ is the completion of $\mathbf{V}$ with respect to $\|\cdot\|$. Since the norm $\|\cdot\|$ is derived via (8.9), it is easy to see that $\|\cdot\|$ is a reasonable and even uniform crossnorm.

Let us consider in $\mathbf{V}_{\|\cdot\|}$ a flow generated by a densely defined operator $A\in L(\mathbf{V}_{\|\cdot\|}, \mathbf{V}_{\|\cdot\|})$. More precisely, there exists a collection of bijective maps $\boldsymbol{\varphi}_t : \mathcal{D}(A)\longrightarrow \mathcal{D}(A)$, here $\mathcal{D}(A)$ denotes the domain of A, satisfying

(i) $\boldsymbol{\varphi}_0 = \mathbf{id}$,

(ii) $\boldsymbol{\varphi}_{t+s} = \boldsymbol{\varphi}_t \circ \boldsymbol{\varphi}_s$, and

(ii) for $\mathbf{u}_0 \in \mathcal{D}(A)$, the map $t \mapsto \boldsymbol{\varphi}_t$ is differentiable as a curve in $\mathbf{V}_{\|\cdot\|}$, and $\mathbf{u}(t) := \boldsymbol{\varphi}_t(\mathbf{u}_0)$ satisfies

$$\dot{\mathbf{u}} = A\mathbf{u}, \tag{8.10}$$

$$\mathbf{u}(0) = \mathbf{u}_0. \tag{8.11}$$

In this framework we want to study the approximation of a solution $\mathbf{u}(t) = \boldsymbol{\varphi}_t(\mathbf{u}_0) \in \mathbf{V}_{\|\cdot\|}$ by a curve $\mathbf{v}_r(t) := \lambda(t) \otimes_{j=1}^d v_j(t)$ in the Hilbert manifold $\mathcal{M}_{(1,\ldots,1)}(\mathbf{V})$, also called in [19] the *Hartree manifold.* The time–dependent Hartree method consists in the use of the Dirac–Frenkel variational principle on the Hartree manifold. More precisely, we want to solve the following Reduced Order Model:

$$\dot{\mathbf{v}}_r(t) = \mathcal{P}_{\mathbf{v}_r(t)}(A\mathbf{v}_r(t)) \text{ for } t \in I, \tag{8.12}$$

$$\mathbf{v}_r(0) = \mathbf{v}_0, \tag{8.13}$$

with $\mathbf{v}_0 = \lambda_0 \otimes_{j=1}^d v_0^{(j)} \in \mathcal{M}_{(1,\ldots,1)}(\mathbf{V})$ being an approximation of $\mathbf{u}_0$[7]. From Corollary 8.4, for each $t > 0$ we would like to find $\dot{\mathbf{v}}_r(t) \in \mathrm{T}_{\mathbf{v}_r(t)} i \left(\mathbb{T}_{\mathbf{v}_r(t)}(\mathcal{M}_{(1,\ldots,1)}(\mathbf{V}))\right)$ such that

$$\langle \dot{\mathbf{v}}_r(t) - A\mathbf{v}_r(t), \dot{\mathbf{v}}(t)\rangle = 0 \text{ for all } \dot{\mathbf{v}}(t) \in \mathrm{T}_{\mathbf{v}_r(t)} i \left(\mathbb{T}_{\mathbf{v}_r(t)}(\mathcal{M}_{(1,\ldots,1)}(\mathbf{V}))\right), \tag{8.14}$$

$$\mathbf{v}_r(0) = \mathbf{v}_0 = \lambda_0 \otimes_{j=1}^d v_0^{(j)}, \tag{8.15}$$

and where, without loss of generality, we may assume $\|v_0^{(j)}\|_j = 1$ for $1 \le j \le d$. A first result is the following.

Lemma 8.5 *Let $\mathbf{v} \in \mathcal{C}^1(I, \mathcal{U}(\mathbf{v}_0))$, where $\mathbf{v}(0) = \mathbf{v}_0 \in \mathcal{M}_{(1,\ldots,1)}(\mathbf{V})$ and $(\mathcal{U}(\mathbf{v}_0), \Theta_{\mathbf{v}_0})$ is a local chart for $\mathbf{v}_0$ in $\mathcal{M}_{(1,\ldots,1)}(\mathbf{V})$. Assume that $\mathbf{v}$ is also a $\mathcal{C}^1$-morphism between the manifolds $I \subset \mathbb{R}$ and $\mathcal{U}(\mathbf{v}_0) \subset \mathcal{M}_{(1,\ldots,1)}(\mathbf{V})$ such that $\mathbf{v}(t) = \lambda(t) \bigotimes_{j=1}^d v_j(t)$ for some $\lambda \in \mathcal{C}^1(I, \mathbb{R})$ and $v_j \in \mathcal{C}^1(I, V_j)$ for $1 \le j \le d$. Then*

$$\dot{\mathbf{v}}(t) = \dot{\lambda}(t) \bigotimes_{j=1}^d v_j(t) + \lambda(t) \sum_{j=1}^d \dot{v}_j(t) \otimes \bigotimes_{k \neq j} v_k(t) = \mathrm{T}_{\mathbf{v}(t)} i(\mathrm{T}_t \mathbf{v}(1)). \tag{8.16}$$

Moreover, if $v_j(t) \in \mathbb{S}_{V_j}$, i.e., $\|v_j(t)\|_j = 1$, for $t \in I$ and $1 \le j \le d$, then $\dot{v}_j(t) \in \mathbb{T}_{v_j(t)}(\mathbb{S}_{V_j})$ for $t \in I$ and $1 \le j \le d$.

Proof. First at all, we recall that by the construction of $\mathcal{U}(\mathbf{v}_0)$ it follows that $W_j^{\min}(\mathbf{v}_0) = W_j^{\min}(\mathbf{v}(t))$ and that $U_j^{\min}(\mathbf{v}_0) = \operatorname{span}\{v_0^{(j)}\}$ is linearly isomorphic to $U_j^{\min}(\mathbf{v}(t))$ for all $t \in I$ and $1 \le j \le d$. Assume $\Theta_{\mathbf{v}_0}(\mathbf{v}(t)) = (\lambda(t), L_1(t), \ldots, L_d(t))$, i.e.,

$$\mathbf{v}(t) := \lambda(t) \bigotimes_{j=1}^d (id_j + L_j(t))\,(v_0^{(j)}),$$

where $\lambda \in \mathcal{C}^1(I, \mathbb{R} \setminus \{0\})$, $L_j \in \mathcal{C}^1(I, \mathcal{L}(U_j^{\min}(\mathbf{v}_0), W_j^{\min}(\mathbf{v}_0)))$ and $(id_j + L_j(t))(v_0^{(j)}) \in U_j^{\min}(\mathbf{v}(t))$ for $1 \le j \le d$. We point out that the linear map $\mathrm{T}_t\mathbf{v} : \mathbb{R} \to \mathbb{T}_{\mathbf{v}(t)}(\mathcal{M}_{(1,\ldots,1)}(\mathbf{V}))$ is characterised by

$$\mathrm{T}_t\mathbf{v}(1) = (\Theta_{\mathbf{v}_0} \circ \mathbf{v})'(t) = (\dot{\lambda}(t), \dot{L}_1(t), \ldots, \dot{L}_d(t)). \tag{8.17}$$

Since $L_j \in \mathcal{C}^1(I, \mathcal{L}(U_j^{\min}(\mathbf{v}_0), W_j^{\min}(\mathbf{v}_0)))$ then $\dot{L}_j \in \mathcal{C}^0(I, \mathcal{L}(U_j^{\min}(\mathbf{v}_0), W_j^{\min}(\mathbf{v}_0)))$. Observe that $U_j^{\min}(\mathbf{v}_0)$ and $U_j^{\min}(\mathbf{v}(t))$ have $W_j^{\min}(\mathbf{v}_0)$ as a common complement. From Lemma 2.5 we known that

$$P_{U_j^{\min}(\mathbf{v}_0) \oplus W_j^{\min}(\mathbf{v}_0)}|_{U_j^{\min}(\mathbf{v}(t))} : U_j^{\min}(\mathbf{v}(t)) \longrightarrow U_j^{\min}(\mathbf{v}_0)$$

is a linear isomorphism. We can write

$$L_j(t) = L_j(t) P_{U_j^{\min}(\mathbf{v}_0) \oplus W_j^{\min}(\mathbf{v}_0)} \text{ and } \dot{L}_j(t) = \dot{L}_j(t) P_{U_j^{\min}(\mathbf{v}_0) \oplus W_j^{\min}(\mathbf{v}_0)},$$

[7]Indeed, $\mathbf{v}_0$ can be chosen as the best approximation of $\mathbf{u}_0$ in $\mathcal{M}_{(1,\ldots,1)}(\mathbf{V})$ because $\mathcal{M}_{(1,\ldots,1)}(\mathbf{V}) = \mathcal{T}_{(1,\ldots,1)}(\mathbf{V}) \setminus \{\mathbf{0}\}$.

and then in (8.17) we identify $\dot{L}_j(t) \in \mathcal{L}(U_j^{\min}(\mathbf{v}_0), W_j^{\min}(\mathbf{v}_0)))$ with

$$\dot{L}_j(t) P_{U_j^{\min}(\mathbf{v}_0)\oplus W_j^{\min}(\mathbf{v}_0)}|_{U_j^{\min}(\mathbf{v}(t))} \in \mathcal{L}(U_j^{\min}(\mathbf{v}(t)), W_j^{\min}(\mathbf{v}_0))).$$

Introduce $v_j(t) := (id_j + L_j(t))(v_0^{(j)})$ for $1 \leq j \leq d$. Then

$$\dot{L}_j(t)(v_j(t)) = \dot{L}_j(t) P_{U_j^{\min}(\mathbf{v}_0)\oplus W_j^{\min}(\mathbf{v}_0)}|_{U_j^{\min}(\mathbf{v}(t))}(v_0^{(j)} + L_j(t)(v_0^{(j)})) = \dot{L}_j(t)(v_0^{(j)})$$

holds for all $t \in I$ and $1 \leq j \leq d$. Hence

$$\dot{v}_j(t) = \dot{L}_j(t)(v_0^{(j)}) = \dot{L}_j(t)(v_j(t)) \tag{8.18}$$

holds for all $t \in I$ and $1 \leq j \leq d$. From Lemma 7.3(b) and (8.17), we have

$$\mathrm{T}_{\mathbf{v}(t)} i(\mathrm{T}_t \mathbf{v}(1)) = \dot{\lambda}(t) \bigotimes_{j=1}^d v_j(t) + \lambda(t) \sum_{j=1}^d \dot{L}_j(t)(v_j(t)) \otimes \bigotimes_{k \neq j} v_k(t),$$

and, by using (8.18) for $\mathbf{v}(t) = \lambda(t) \bigotimes_{j=1}^d v_j(t)$, we obtain (8.16).

To prove the second statement, recall that $U_j^{\min}(\mathbf{v}(t)) = \mathrm{span}\,\{v_j(t)\}$ and $V_j = U_j^{\min}(\mathbf{v}(t)) \oplus W_j^{\min}(\mathbf{v}_0)$ for $1 \leq j \leq d$. Then we consider

$$W_j^{\min}(\mathbf{v}_0) = \mathrm{span}\,\{v_j(t)\}^\perp = \{u_j \in V_j : \langle u_j, v_j(t)\rangle_j = 0\} \text{ for } 1 \leq j \leq d,$$

and hence $\langle \dot{v}_j(t)), v_j(t)\rangle_j = 0$ holds for $1 \leq j \leq d$. From Remark 2.16, we have $(\dot{v}_1(t), \ldots, \dot{v}_d(t)) \in \mathcal{C}(I, \times_{j=1}^d \mathbb{T}_{v_j(t)}(\mathbb{S}_{V_j}))$, because of $W_j^{\min}(\mathbf{v}_0) = \mathbb{T}_{v_j(t)}(\mathbb{S}_{V_j})$ for $1 \leq j \leq d$. ∎

Before stating the next result, we introduce for $\mathbf{v}_r(t) = \lambda(t) \bigotimes_{j=1}^d v_j(t)$ the following time dependent bilinear forms

$$\mathrm{a}_k(t; \cdot, \cdot) : V_k \times V_k \longrightarrow \mathbb{R},$$

by

$$\mathrm{a}_k(t; z_k, y_k) := \left\langle A\left(z_k \otimes \bigotimes_{j \neq k} v_j(t)\right), \left(y_k \otimes \bigotimes_{j\neq k} v_j(t)\right)\right\rangle$$

for each $1 \leq k \leq d$. Now, we will show the next result (compare with Theorem 3.1 in [19]).

Theorem 8.6 (Time dependent Hartree method) *The solution $\mathbf{v}_r(t) = \lambda(t) \bigotimes_{j=1}^d v_j(t)$ for $(v_1(t), \ldots, v_d(t)) \in \times_{j=1}^d \mathbb{S}_{V_j}$ of*

$$\dot{\mathbf{v}}_r(t) = \mathcal{P}_{\mathbf{v}_r(t)}(A\mathbf{v}_r(t)) \text{ for } t \in I, \tag{8.19}$$

$$\mathbf{v}_r(0) = \mathbf{v}_0, \tag{8.20}$$

satisfies

$$\langle \dot{v}_j(t), \dot{w}_j(t)\rangle_j - \mathrm{a}_j(t; v_j(t), \dot{w}_j(t)) = 0 \quad \text{for all } \dot{w}_j(t) \in \mathbb{T}_{v_j(t)}(\mathbb{S}_{V_j}), \quad 1 \leq j \leq d,$$

and

$$\lambda(t) = \lambda_0 \exp\left(\int_0^t \left\langle A\left(\otimes_{j=1}^d v_j(s)\right), \otimes_{j=1}^d v_j(s)\right\rangle ds\right).$$

Proof. From Lemma 8.5 we have $\mathbb{T}_{\mathbf{v}_r(t)}\left(\mathcal{M}_{(1,\ldots,1)}(\mathbf{V})\right) = \mathbb{R} \times \times_{j=1}^d \mathbb{T}_{v_j(t)}(\mathbb{S}_{V_j})$, Thus, for each $\dot{\mathbf{w}}(t) \in \mathrm{T}_{\mathbf{v}(t)} i\left(\mathbb{T}_{\mathbf{v}(t)}\left(\mathcal{M}_{(1,\ldots,1)}(\mathbf{V})\right)\right)$ there exists $(\dot{\beta}(t), \dot{w}_1(t), \ldots, \dot{w}_d(t)) \in \mathbb{R} \times \times_{j=1}^d \mathbb{T}_{v_j(t)}(\mathbb{S}_{V_j})$, such that

$$\dot{\mathbf{w}}(t) = \dot{\beta}(t) \bigotimes_{j=1}^d v_j(t) + \lambda(t) \sum_{j=1}^d \dot{w}_j(t) \otimes \bigotimes_{k\neq j} v_k(t).$$

Then (8.14) holds if and only if

$$\left\langle \dot{\mathbf{v}}_r(t) - A\mathbf{v}_r(t), \dot{\beta}(t)\bigotimes_{j=1}^{d} v_j(t) + \lambda(t)\sum_{j=1}^{d}\dot{w}_j(t)\otimes\bigotimes_{k\neq j} v_k(t)\right\rangle = 0$$

for all $(\dot{\beta}(t), \dot{w}_1(t), \ldots, \dot{w}_d(t)) \in \mathbb{R}\times \bigtimes_{j=1}^{d}\mathbb{T}_{v_j(t)}(\mathbb{S}_{V_j})$. Then

$$\dot{\lambda}(t)\dot{\beta}(t) + \lambda(t)^2\sum_{j=1}^{d}\left(\langle \dot{v}_j(t), \dot{w}_j(t)\rangle_j - \langle A\bigotimes_{s=1}^{d} v_s(t), \dot{w}_j(t)\otimes\bigotimes_{k\neq j} v_k(t)\rangle\right)$$
$$-\lambda(t)\dot{\beta}(t)\langle A\bigotimes_{j=1}^{d} v_j(t), \bigotimes_{j=1}^{d} v_j(t)\rangle = 0,$$

i.e.,

$$\dot{\beta}(t)\left(\dot{\lambda}(t) - \lambda(t)\langle A\bigotimes_{j=1}^{d} v_j(t), \bigotimes_{j=1}^{d} v_j(t)\rangle\right) \tag{8.21}$$

$$+\lambda(t)^2\sum_{j=1}^{d}\left(\langle \dot{v}_j(t), \dot{w}_j(t)\rangle_j - \langle A\bigotimes_{s=1}^{d} v_s(t), \dot{w}_j(t)\otimes\bigotimes_{k\neq j} v_k(t)\rangle\right) = 0 \tag{8.22}$$

holds for all $\dot{\beta}(t)\in\mathbb{R}$ and $(\dot{w}_1(t), \ldots, \dot{w}_d(t)) \in \bigtimes_{j=1}^{d}\mathbb{T}_{v_j(t)}(\mathbb{S}_{V_j})$. If $\lambda(t)$ solves the differential equation

$$\dot{\lambda}(t) = \left\langle A\left(\otimes_{j=1}^{d} v_j(t)\right), \otimes_{j=1}^{d} v_j(t)\right\rangle\lambda(t) \tag{8.23}$$

$$\lambda(0) = \lambda_0, \tag{8.24}$$

i.e.,

$$\lambda(t) = \lambda_0\exp\left(\int_0^t \left\langle A\left(\otimes_{j=1}^{d} v_j(s)\right), \otimes_{j=1}^{d} v_j(s)\right\rangle ds\right),$$

then the first term of (8.22) is equal to 0. Therefore, from (8.22) we obtain that for all $j\in D$,

$$\langle \dot{v}_j(t), \dot{w}_j(t)\rangle_j - \langle A\bigotimes_{s=1}^{d} v_s(t), \dot{w}_j(t)\otimes\bigotimes_{k\neq j} v_k(t)\rangle = 0,$$

that is,

$$\langle \dot{v}_j(t), \dot{w}_j(t)\rangle_j - \mathrm{a}_j(t; v_j(t), \dot{w}_j(t)) = 0$$

holds for all $\dot{w}_j(t)\in\mathbb{T}_{v_j(t)}(\mathbb{S}_{V_j})$, and the theorem follows. ■

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