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The Set of Periods for a Class of Crazy Maps

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The crazy maps are a class of continuous maps from $\Sigma_N \times \mathbb{S}^1$, where Σ_N is the product space of the bi-infinite sequences on N symbols and $\mathbb{S}^1$ is the unit circle, into itself. Moreover, each of these maps has N orientation-preserving circle homeomorphisms associated with it. In this paper we study the set of periods in the case $N = 2$ and where the associated maps are rotations. © 1998 Academic Press

1. INTRODUCTION

In 1974 Afraimovich and Shilnikov [1] described the semi-hyperbolic invariant set generated by a bifurcation of several homoclinic surfaces of a saddle-node cycle. The invariant set in the last bifurcation is homeomorphic to the product space $\Sigma_N \times \mathbb{S}^1$, where $\Sigma_N = \{0, 1, \dots, N-1\}^{\mathbb{Z}}$ is the space of all bi-infinite sequences

$$\underline{a} = \{\dots a_{-n} \dots a_{-1}.a_0 a_1 \dots a_n \dots\}$$

of symbols $0, 1, \dots, N-1$ (we note that in this paper we shall use the same notation as in [2]). The dynamics on the invariant set above after some rescaling, gives rise to *crazy dynamics*. It is described as follows.

Let $\underline{a} = \{\dots a_{-1}.a_0 a_1 \dots\} \in \Sigma_N$. Then $\sigma : \Sigma_N \to \Sigma_N$, *the shift map,* is given by

$$\sigma(\underline{a}) = \{\dots a_{-1} a_0 . a_1 a_2 \dots\}.$$

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Let $f_0, f_1, \ldots, f_{N-1} : \mathbb{S}^1 \to \mathbb{S}^1$, N orientation-preserving circle homeomorphisms. Then the map $\Phi : \Sigma_N \times \mathbb{S}^1 \to \Sigma_N \times \mathbb{S}^1$ given by

$$\Phi(\underline{a}, x) = \big(\sigma(\underline{a}), f_{a_0}(x)\big)$$

is called *the crazy map associated with* $f_0, f_1, \ldots, f_{N-1}$. Note that the nth iterate of this map at the point $(\underline{a}, x) \in \Sigma_N \times \mathbb{S}^1$ is given by

$$\Phi^n(\underline{a}, x) = \big(\sigma^n(\underline{a}), \big(f_{a_{n-1}} \circ \cdots \circ f_{a_1} \circ f_{a_0}\big)(x)\big).$$

The goal of this paper is to characterize the set of periods for a particular class of crazy maps. More precisely, we consider the case when $N = 2$ and f_0 and f_1 are rotations.

The paper is organized as follows. In the next section we give some basic definitions and state the main result of this paper which gives the characterization of the set of periods for crazy maps given by two rotations. Finally, in Section 3 the theorem stated in Section 2 will be proved.

2. DEFINITIONS AND STATEMENT OF THE MAIN RESULT

In this section we will define the crazy maps associated with two rotations. Before stating the main result of this paper we will introduce some preliminary definitions in order to describe the set of periods for the class of maps under consideration.

For each $(\alpha_0, \alpha_1) \in \mathbb{R}^2$ the map $\Phi_{\alpha_0, \alpha_1} : \Sigma_2 \times \mathbb{R} \to \Sigma_2 \times \mathbb{R}$ given by $\Phi_{\alpha_0, \alpha_1}(\underline{a}, x) = (\sigma(\underline{a}), F_{a_0}(x))$, where

$$F_{a_0}(x) = \begin{cases} x + \alpha_0 & \text{if } a_0 = 0, \\ x + \alpha_1 & \text{if } a_0 = 1, \end{cases}$$

will be called *the* (α_0, α_1)*-crazy map*. Note that instead of working with the circle maps themselves we will rather use their liftings to the universal covering space $\mathbb{R}$. Moreover, since $(\alpha_0, \alpha_1) \in \mathbb{R}^2$, we can consider the whole plane as the bifurcation space for this class of maps.

Let $\underline{a} = \{\ldots a_{-1}.a_0 a_1 \ldots\}$ and $\underline{b} = \{\ldots b_{-1}.b_0 b_1 \ldots\}$ be two elements of Σ_2. We define a metric d given by

$$d(\underline{a}, \underline{b}) = \sum_{i=-\infty}^{\infty} \frac{\gamma(a_i, b_i)}{2^{|i|}},$$

where

$$\gamma(a_i, b_i) = \begin{cases} 0 & \text{if } a_i = b_i, \\ 1 & \text{if } a_i \neq b_i. \end{cases}$$

Then Σ_2 is a Hausdorff compact space and the map σ is a homeomorphism (see [2]). We consider $\Sigma_2 \times \mathbb{R}$ endowed with the product topology. Thus, the (α_0, α_1)-crazy maps are a class of homeomorphisms from $\Sigma_2 \times \mathbb{R}$ into itself.

Next, we will then define the set of periods of a (α_0, α_1)-crazy map. Let $(\alpha_0, \alpha_1) \in \mathbb{R}^2$ and $n \in \mathbb{N}$. Then $n \in \operatorname{Per}(\Phi_{\alpha_0, \alpha_1})$ if there exists $(\underline{a}, x) \in \Sigma_2 \times \mathbb{R}$ such that $\Phi^n_{\alpha_0, \alpha_1}(\underline{a}, x) \in \{\underline{a}\} \times \{x + \mathbb{Z}\}$ and $\Phi^j_{\alpha_0, \alpha_1}(\underline{a}, x) \notin \{\underline{a}\} \times \{x + \mathbb{Z}\}$ for $j = 1, 2, \ldots, n - 1$. In this case we will say that $(\underline{a}, x)$ is a *periodic point of* $\Phi_{\alpha_0, \alpha_1}$ *of period* n.

Let $(\alpha_0, \alpha_1), (\alpha'_0, \alpha'_1) \in \mathbb{R}^2$, We will say that the maps $\Phi_{\alpha_0, \alpha_1}$ and $\Phi_{\alpha'_0, \alpha'_1}$ are *topologically conjugate* if there exists a homeomorphism $h : \Sigma_2 \times \mathbb{R} \to \Sigma_2 \times \mathbb{R}$, where $h(\underline{a}, x) = (h_1(\underline{a}, x), h_2(\underline{a}, x))$, such that $h_2(\underline{a}, x + 1) = h_2(\underline{a}, x) + 1$ for all $x \in \mathbb{R}$ and $h \circ \Phi_{\alpha_0, \alpha_1} = \Phi_{\alpha'_0, \alpha'_1} \circ h$. Note that if $\Phi_{\alpha_0, \alpha_1}$ and $\Phi_{\alpha'_0, \alpha'_1}$ are topologically conjugate then $\operatorname{Per}(\Phi_{\alpha_0, \alpha_1}) = \operatorname{Per}(\Phi_{\alpha'_0, \alpha'_1})$.

Let $\mathbb{Q}_{[0,1]}$ denote $\mathbb{Q} \cap [0, 1]$.

Let $(\alpha_0, \alpha_1) \in \mathbb{R}^2$. We will say that $(n, p/q) \in \mathbb{N} \times \mathbb{Q}_{[0,1]}$, with $(p, q) = 1$, is (α_0, α_1)-*admissible* if one of the following conditions hold.

(C1) $n = 1$, $p/q = 0$ (respectively, $p/q = 1$) and $(\alpha_0, \alpha_1) \in \mathbb{Z} \times \mathbb{R}$ (respectively, $(\alpha_0, \alpha_1) \in \mathbb{R} \times \mathbb{Z}$).

(C2) $n > 1$, $n = qms$ for some m, $s \in \mathbb{N}$, with $m \neq q$, and where

$$\frac{jn}{mq}\big(\alpha_0(q - p) + \alpha_1 p\big) \notin \mathbb{Z}$$

for $j = 1, 2, \ldots, m - 1$ and

$$\frac{n}{q}\big(\alpha_0(q - p) + \alpha_1 p\big) \in \mathbb{Z}.$$

Let Δ denote the set of pairs $(\alpha_0, \alpha_1) \in \mathbb{R}^2$ such that $\alpha_0 < \alpha_1$ and $\alpha_1 \neq -\alpha_0$. Then the main result of this paper is the following.

THEOREM 2.1. *Let* $(\alpha_0, \alpha_1) \in \mathbb{R}^2$. *Then the following statements hold.*

(a) *The maps* $\Phi_{\alpha_0, \alpha_1}$ *and* $\Phi_{\alpha_1, \alpha_0}$ *are topologically conjugate.*

(b) *Assume that* $\alpha_1 = |\alpha_0|$. *Then the following statements hold.*

(b.1) *If* $\alpha_1 = p/q$, *with* $(p, q) = 1$, *then* $\operatorname{Per}(\Phi_{\alpha_1, \alpha_1}) = q\mathbb{N}$ *and* $\operatorname{Per}(\Phi_{-\alpha_1, \alpha_1}) = 2\mathbb{N} + q\mathbb{Z}^+$.

(b.2) *If* $\alpha_1 \notin \mathbb{Q}$ *then* $\operatorname{Per}(\Phi_{\alpha_1, \alpha_1}) = \emptyset$ *and* $\operatorname{Per}(\Phi_{-\alpha_1, \alpha_1}) = 2\mathbb{N}$.

(c) *Assume that* $(\alpha_0, \alpha_1) \in \Delta$. *If* $\alpha_0 = p_0/q_0 \in \mathbb{Q}$, *with* $(p_0, q_0) = 1$ (*respectively*, $\alpha_0 \notin \mathbb{Q}$) *and* $\alpha_1 \notin \mathbb{Q}$ (*respectively*, $\alpha_1 = p_1/q_1 \in \mathbb{Q}$, *with* $(p_1, q_1) = 1$) *then* $\mathrm{Per}(\Phi_{\alpha_0, \alpha_1}) = \{q_0\}$ (*respectively*, $\mathrm{Per}(\Phi_{\alpha_0, \alpha_1}) = \{q_1\}$). *Otherwise*, $n \in \mathrm{Per}(\Phi_{\alpha_0, \alpha_1})$ *if and only if there exist* $p/q \in \mathbb{Q}_{[0,1]}$, *with* $(p, q) = 1$, *such that* $(n, p/q)$ *is* (α_0, α_1)*-admissible*.

The rest of this paper is devoted to the proof of Theorem 2.1.

3. PROOF OF THEOREM 2.1

We start this section by proving the first statement of Theorem 2.1.

Proof of Theorem 2.1(a). To prove it, for $\underline{a} = \cdots a_{-1}a_0a_1 \cdots \in \Sigma_2$, we take $g(\underline{a}) = \cdots(1 - a_{-1})(1 - a_0)(1 - a_1)\cdots$. Clearly, $g : \Sigma_2 \to \Sigma_2$ is a homeomorphism and $\sigma \circ g = g \circ \sigma$. Let $h : \Sigma_2 \times \mathbb{R} \to \Sigma_2 \times \mathbb{R}$ given by $h(\underline{a}, x) = (g(\underline{a}), x)$. Obviously, h is also a homeomorphism and

$$\begin{aligned} h\big(\Phi_{\alpha_0, \alpha_1}((\underline{a}, x))\big) &= h\big(\sigma(\underline{a}), F_{a_0}(x)\big) \\ &= \big(g(\sigma(\underline{a})), F_{a_0}(x)\big) \\ &= \big(\sigma(g(\underline{a})), F_{a_0}(x)\big) \\ &= \Phi_{\alpha_1, \alpha_0}(g(\underline{a}), x) \\ &= \Phi_{\alpha_1, \alpha_0}(h(\underline{a}, x)). \end{aligned}$$

This ends the proof of statement (a). ∎

Before proving statements (b) and (c) we give some preliminary definitions and results.

It is not difficult to see that if $\underline{a}$ is periodic of period n then we can write

$$\begin{aligned} \underline{a} &= \{\ldots(a_{-n}\ldots a_{-1}).(a_0a_1a_2\ldots a_{n-1})\ldots\} \\ &= \{\ldots(b_0\ldots b_{n-1}).(b_0\ldots b_{n-1})\ldots\} \\ &= \{\overline{(b_0\ldots b_{n-1}).(b_0\ldots b_{n-1})}\}, \end{aligned}$$

where the bi-infinite sequences which are periodically repeated after some fixed length are denoted by the finite length sequence with an overbar.

Now, take $\underline{a} = \{\ldots a_{-1}.a_0a_1\ldots\} \in \Sigma_2$. Then the real number

$$\lim_{m\to\infty} \frac{1}{m} \sum_{j=0}^{m-1} a_j$$

will be denoted by $\mu^+(\underline{a})$ if it exists.

LEMMA 3.1. *The following statements hold.*

(a) *Let* $\underline{a} \in \Sigma_2$ *be a periodic sequence of period* n. *Then* $\mu^+(\underline{a})$ *exists, and is rational.*

(b) *Let* $p/q \in \mathbb{Q}_{[0,1]}$, *with* $(p,q) = 1$. *Then for each* $s \in \mathbb{N}$ *there exists* $\underline{a} \in \Sigma_2$ *periodic of period* sq *such that* $\mu^+(\underline{a}) = p/q$.

Proof. Assume that $\underline{a} = \{\dots a_{-1}.a_0 a_1 \dots\} \in \Sigma_2$ is periodic of period n. Then, we can write

$$\underline{a} = \{\overline{(b_0 \dots b_{n-1}).(b_0 \dots b_{n-1})}\}.$$

Suppose that $(1/n)\sum_{j=0}^{n-1} b_j = p/q \in \mathbb{Q}_{[0,1]}$, with $(p,q) = 1$. Then

$$\begin{aligned}\lim_{m\to\infty} \frac{1}{m} \sum_{j=0}^{m-1} a_j &= \lim_{ns\to\infty} \frac{1}{ns}\left(\overbrace{\sum_{j=0}^{n-1} b_j + \cdots + \sum_{j=0}^{n-1} b_j}^{s}\right)\\ &= \lim_{ns\to\infty} \frac{1}{s}\left(s\frac{p}{q}\right)\\ &= \frac{p}{q}.\end{aligned}$$

This ends the proof of statement (a). To prove (b), assume that $p/q \in \mathbb{Q}_{[0,1]}$, with $(p,q) = 1$, and $s \in \mathbb{N}$. Then set

$$\epsilon(p/q) = \begin{cases} \overbrace{0\cdots 0}^{q-p-1}\ \overbrace{1\cdots 1}^{p-1} & \text{if } q > 2,\\ \text{the empty word} & \text{if } q = 2,\end{cases}$$

and define the sequence

$$\underline{a} = \{\overline{0\epsilon(p/q)01\epsilon(p/q)01\dots01\epsilon(p/q)1.\underbrace{0\epsilon(p/q)01\epsilon(p/q)01\dots01\epsilon(p/q)1}_{sq}}\}.$$

By construction $\underline{a}$ is periodic of period sq and $\mu^+(\underline{a}) = p/q$. This follows statement (b). ∎

We observe that the nth iterate of a given (α_0, α_1)-crazy map at the point $(\underline{a}, x) \in \Sigma_N \times \mathbb{S}^1$ is given by

$$\Phi^n_{\alpha_0,\alpha_1}(\underline{a}, x) = \left(\sigma^n(\underline{a}), x + \alpha_0 \lambda_0^n(\underline{a}) + \alpha_1 \lambda_1^n(\underline{a})\right), \tag{3.1}$$

where

$$\lambda_0^n(\underline{a}) = \sum_{j=0}^{n-1} (1 - a_j) \qquad \text{and} \qquad \lambda_1^n(\underline{a}) = \sum_{j=0}^{n-1} a_j.$$

Finally, note that $\lambda_0^n(\underline{a})$ (respectively, $\lambda_1^n(\underline{a})$) gives the number of 0's (respectively, 1's) in the finite set $\{a_0, \dots, a_{n-1}\}$.

LEMMA 3.2. *For each $n \in \mathbb{N}$ the following statements hold.*

(a) *There exists $\underline{a} \in \Sigma_2$ periodic of period n.*

(b) *For each $i, j \in N$ such that $i + j = n$ there exists $\underline{a} \in \Sigma_2$ periodic of period n satisfying that $\lambda_0^n(\underline{a}) = i$ and $\lambda_1^n(\underline{a}) = j$.*

Proof. Statements (a) and (b) follow respectively if we take, for each $n \in \mathbb{N}$, the sequences

$$\underline{a} = \{\overline{0 \cdots 01.\underbrace{0 \cdots 01}_{n}}\}$$

and

$$\underline{a}^* = \{\overline{0 \cdots 01 \cdots 1.\underbrace{0 \cdots 0}_{i}\,\underbrace{1 \cdots 1}_{j}}\},$$

where $i + j = n$, respectively. ∎

Proof of Theorem 2.1(b). Assume first that $\alpha_1 = p/q \in \mathbb{Q}$, with $(p, q) = 1$. Assume that $n \in \operatorname{Per}(\Phi_{\alpha_1, \alpha_1})$ and let $(\underline{a}, x)$ be a periodic point of $\Phi_{\alpha_1, \alpha_1}$ of period n. Then, $\sigma^n(\underline{a}) = \underline{a}$ and the second component of $\Phi^n_{\alpha_1, \alpha_1}(\underline{a}, x)$ is equal to

$$(F_{a_{n-1}} \circ \cdots \circ F_{a_1} \circ F_{a_0})(x) = x + n\frac{p}{q}.$$

Since $n(p/q) \in \mathbb{Z}$ and $(p, q) = 1$ we have that $n = mq$ for some $m \in \mathbb{N}$. Thus, $\operatorname{Per}(\Phi_{\alpha_1, \alpha_1}) \subset q\mathbb{N}$. Now, let $n \in q\mathbb{N}$. Then there exists $m \in \mathbb{N}$. Such that $n = mq$. Using Lemma 3.1 we construct $\underline{a}$ a periodic sequence of period n. From all said above $(\underline{a}, x)$ is a periodic point of period n. Thus, $\operatorname{Per}(\Phi_{\alpha_1, \alpha_1}) = q\mathbb{N}$ and the first part of statement (b.1) follows.

Now, assume that $n \in \operatorname{Per}(\Phi_{-\alpha_1, \alpha_1})$ and let $(\underline{a}, x)$ be a period point of $\Phi_{-\alpha_1, \alpha_1}$ of period n. From (3.1) we have that

$$(F_{a_{n-1}} \circ \cdots \circ F_{a_1} \circ F_{a_0})(x) = x + \frac{p}{q}(\lambda_1^n(\underline{a}) - \lambda_0^n(\underline{a})),$$

where

$$n = \lambda_1^n(\underline{a}) + \lambda_0^n(\underline{a}). \tag{3.2}$$

Since $(p/q)(\lambda_1^n(\underline{a}) - \lambda_0^n(\underline{a})) \in \mathbb{Z}$ and $(p, q) = 1$ we have that

$$mq = |\lambda_1^n(\underline{a}) - \lambda_0^n(\underline{a})| \tag{3.3}$$

for some $m \in \mathbb{N}$. Then

$$n = \begin{cases} 2\lambda_0^n(\underline{a}) + mq & \text{if } \lambda_1^n(\underline{a}) > \lambda_0^n(\underline{a}), \\ 2\lambda_1^n(\underline{a}) + mq & \text{if } \lambda_1^n(\underline{a}) < \lambda_0^n(\underline{a}), \\ 2\lambda_1^n(\underline{a}) & \text{if } \lambda_1^n(\underline{a}) = \lambda_0^n(\underline{a}), \end{cases}$$

and $\operatorname{Per}(\Phi_{-\alpha_1, \alpha_1}) \subset 2\mathbb{N} + q\mathbb{Z}^+$.

Let $n \in 2\mathbb{N} + q\mathbb{Z}^+$. Then $n = 2k + qm$ for some $k \in \mathbb{N}$ and $m \in \mathbb{Z}^+$. Assume first that $m \neq 0$. If $m = 0$ we proceed in a similar way. We choose $i = k$ and $j = n - k$. Then by using Lemma 3.2(b) we construct a periodic sequence $\underline{a}$ of period n satisfying that $\lambda_0^n(\underline{a}) = i$ and $\lambda_1^n(\underline{a}) = j$. Clearly, $(\underline{a},x)$ is a periodic point of $\Phi_{-\alpha_1, \alpha_1}$ of period n satisfying (3.2) and (3.3). Thus, we have proved that $\operatorname{Per}(\Phi_{-\alpha_1, \alpha_1}) \supset 2\mathbb{N} + q\mathbb{Z}^+$. This ends the proof of statement (b.1). Statement (b.2) follows using some similar arguments. ∎

Proof of Theorem 2.1(c). Assume that $(\alpha_0, \alpha_1) \in \Delta$. Let $\alpha_0 = p_0/q_0 \in \mathbb{Q}$, with $(p_0, q_0) = 1$ and $\alpha_1 \notin \mathbb{Q}$. Let $n \in \operatorname{Per}(\Phi_{\alpha_1, \alpha_1})$ and let $(\underline{a}, x)$ be a periodic point of $\Phi_{\alpha_0, \alpha_1}$ of period n. Thus, $\Phi^n_{\alpha_0, \alpha_1}(\underline{a}, x) \in \{\underline{a}\} \times \mathbb{Z}$ and $\Phi^j_{\alpha_0, \alpha_1}(\underline{a}, x) \notin \{\underline{a}\} \times \mathbb{Z}$ for $j = 1, 2, \dots, n-1$. Assume that $\underline{a}$ is periodic of period q and $\mu^+(\underline{a}) = p/q$. Then $n = mq$ for some $m \in \mathbb{N}$. Moreover, by using (3.1) we have that

$$\frac{p_0}{q_0} j(q-p) + \alpha_1 jp \notin \mathbb{Z}$$

for $j = 1, 2, \dots, m-1$ and

$$\frac{p_0}{q_0} m(q-p) + \alpha_1 mp = k$$

for some $k \in \mathbb{Z}$. Then $p = 0$ and $q = 1$, otherwise

$$\alpha_1 = \frac{(p_0/q_0)mq - (p_0/q_0)mp - k}{mp} \in \mathbb{Q},$$

a contradiction. Thus, since $\underline{a}$ is a fixed point of σ and $\mu^+(\underline{a}) = 0$, we obtain that $\underline{a} = \{\overline{0.0}\}$. From above

$$\frac{p_0}{q_0} j \notin \mathbb{Z}$$

for $j = 1, 2, \dots, m-1$ and

$$\frac{p_0}{q_0} m = k.$$

Since $(p_0, q_0) = 1$ we have that $n = m = q_0$. On the other hand, from a direct computation, it follows that $(\{\overline{0.0}\}, x)$ is a periodic point of period q_0.

Now, assume that $n \in \text{Per}(\Phi_{\alpha_0, \alpha_1})$. Let $(\underline{a}, x) \in \Sigma_2 \times \mathbb{R}$ be a periodic point of period n of $\Phi_{\alpha_0, \alpha_1}$. Then $\sigma^n(\underline{a}) = \underline{a}$ and

$$\left(F_{a_{n+1}} \circ \cdots \circ F_{a_1} \circ F_{a_0}\right)(x) = x + k$$

for some $k \in \mathbb{Z}$. If $n = 1$ then, by using that the only fixed points of σ are $\{\overline{0.0}\}$ and $\{\overline{1.1}\}$, we have that

$$k = \begin{cases} \alpha_0 & \text{if } \underline{a} = \{\overline{0.0}\}, \\ \alpha_1 & \text{if } \underline{a} = \{\overline{1.1}\}. \end{cases}$$

Thus, if $k = \alpha_0$ (respectively, $k = \alpha_1$) we take $p/q = 0$ (respectively, $p/q = 1$) and the theorem follows for $n = 1$. Now, assume that $n > 1$. Since $\sigma^n(\underline{a}) = \underline{a}$, then there exists $m \in \mathbb{N}$ such that $n = mr$, $\sigma^r(\underline{a}) = \underline{a}$, and $\sigma^j(\underline{a}) \neq \underline{a}$ for $j = 1, 2, \ldots, r-1$. By using that $(\underline{a}, x)$ is a periodic point of period n and (3.1) we have that

$$\alpha_0 \lambda_0^{jr}(\underline{a}) + \alpha_1 \lambda_1^{jr}(\underline{a}) \notin \mathbb{Z}$$

for $j = 1, 2, \ldots, m-1$ and

$$\alpha_0 \lambda_0^{mr}(\underline{a}) + \alpha_1 \lambda_1^{mr}(\underline{a}) = k \in \mathbb{Z}.$$

From Lemma 3.1(a), $\mu^+(\underline{a}) = p/q \in \mathbb{Q}_{[0,1]}$. Without loss of generality we may assume that $(p, q) = 1$. Then $r = sq$ for some $s \in \mathbb{N}$. Thus, $\lambda_0^r(\underline{a}) = sp$ and $\lambda_0^r(\underline{a}) = s(q-p)$. Finally, we obtain that $n = (ms)q$,

$$\alpha_0 js(q-p) + \alpha_1 jsp = \alpha_0 j \frac{n}{mq}(q-p) + \alpha_1 j \frac{n}{mq} p \notin \mathbb{Z}$$

for $j = 1, 2, \ldots, m-1$ and

$$\begin{aligned} k &= \alpha_0 ms(q-p) + \alpha_1 msp \\ &= \alpha_0 \frac{n}{q}(q-p) + \alpha_1 \frac{n}{q} p. \end{aligned}$$

This ends the proof of the "only if" part.

Now, assume that for a given $n \in \mathbb{N}$ there exist $p/q \in \mathbb{Q}_{[0,1]}$, with $(p, q) = 1$, such that $(n, p/q)$ is (α_0, α_1)-admissible. If $(n, p/q) = (1, 0)$ (respectively, $(n, p/q) = (1, 1)$) and $(\alpha_0, \alpha_1) \in \mathbb{Z} \times \mathbb{R}$ (respectively, $(\alpha_0, \alpha_1) \in \mathbb{Z} \times \mathbb{R}$) we take $(\underline{a}, x) = (\{\overline{0.0}\}, x)$ (respectively, $(\underline{a}, x) =$

$(\{\overline{1.1}\}, x))$ for some $x \in \mathbb{R}$. It is easy to see that in both cases the constructed point is a fixed point of $\Phi_{\alpha_0, \alpha_1}$. Let $n > 1$ and $n = msq$ for some $m, s \in \mathbb{N}$, $m \neq q$. From Lemma 3.1(b) we can construct $\underline{a} \in \Sigma_2$ periodic of period sq such that $\mu^+(\underline{a}) = p/q$. Using some similar arguments as used in the "only if" proof it is not difficult to show that $(\underline{a}, x)$ is a periodic point of period n of $\Phi_{\alpha_0, \alpha_1}$ for all $x \in \mathbb{R}$. This ends the proof of the second statement of Theorem 2.1(c). ∎

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