A Proper Generalized Decomposition for the solution of elliptic problems in abstract form by using a functional Eckart-Young approach

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Abstract

The Proper Generalized Decomposition (PGD) is a methodology initially proposed for the solution of partial differential equations (PDE) defined in tensor product spaces. It consists in constructing a separated representation of the solution of a given PDE. In this paper we consider the mathematical analysis of this framework for a larger class of problems in an abstract setting. In particular, we introduce a generalization of Eckart and Young theorem which allows to prove the convergence of the so-called progressive PGD for a large class of linear problems defined in tensor product Hilbert spaces.

Key words: Proper Generalized Decomposition, Singular Values, Tensor Product Hilbert Spaces.

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1. Introduction

The Proper Generalized Decomposition (PGD) method has been recently proposed [1, 15, 19] for the a priori construction of separated representations of an element u in a tensor product space $V = V_1 \otimes \ldots \otimes V_d$, which is the solution of a problem

$$A(u) = l. \tag{1}$$

A rank-n approximated separated representation u_n of u is defined by

$$u_n = \sum_{i=1}^{n} v_i^1 \otimes \ldots \otimes v_i^d, \tag{2}$$

with $v_i^k \in V_k$ for $1 \leq i \leq n$ and $1 \leq k \leq d$. The a posteriori construction of such tensor decompositions, when the function u is known, have been extensively studied over the past years in multilinear algebra community [6, 7, 13, 14, 4, 8] (essentially for finite dimensional vector spaces V_i). The question of finding an optimal decomposition of a given rank r is not trivial and has led to various definitions and associated algorithms for the separated representations.
In the context of problems of type (1), the solution is not known *a priori*, nor an approximation of it. An approximate solution is even unreachable with traditional numerical techniques when dealing with high dimensions d. It is the so-called curse of dimensionality associated with the dramatic increase of the dimension of approximation spaces when increasing d. The PGD method aims at constructing a decomposition of type (2) without knowing *a priori* the solution u. The aim of the PGD is to construct a sequence u_n based on the knowledge of operator A and right-hand side l. This can be achieved by introducing new definitions of optimal decompositions (2). The Proper Generalized Decomposition (PGD) method have been first introduced under the name of "Radial-type approximation" for the solution of time dependent partial differential equations (PDE), by separating space and time variables, and used in the context of the LATIN method in computational solid mechanics [15, 10, 16, 24, 17, 23]. It has been also introduced for the separation of coordinate in multidimensional PDEs [1, 2], with many applications in kinetic theory of complex fluids, financial mathematics, computational chemistry... It has also been introduced in the context of stochastic or parametrized PDEs by introducing a separation of physical variables (space, time...) and (random) parameters [19, 20, 21]. Still in the context of stochastic PDEs, a further separation of parameters have also been introduced, by exploiting the tensor product structure of stochastic function spaces [9, 22]. In this context, it leads to a representation of functionals of random variables alternative to classical chaos expansions [28, 12, 27, 26, 29]. Of course, separated representations constitute an effective alternative only for functionals of random variables that admit a low rank representation.

Several PGD definitions and associated algorithms have been proposed (see e.g. [20, 23, 5]) and have proved their efficiency in practical applications. However, for most PGD definitions, their mathematical analysis remain open. In this paper, we investigate a particular case of PGD, which consists in defining the decomposition (2) progressively. This is a basic definition of the PGD which was proposed in [15, 19, 1]. A proof of convergence for this particular PGD has been introduced in [18], for the case of a second order elliptic symmetric partial differential equation defined in a 2-dimensional domain, and in [3], for the case of linear systems with a full rank square matrix.

Here, we consider the mathematical analysis of this PGD for a larger class of problems in an abstract setting. We introduce a generalization of Eckart and Young theorem [11] which allows to prove the convergence of progressive PGDs for a large class of linear problems defined in tensor product Hilbert spaces.

The outline of the paper is as follows. In section 2, we introduce the definition of tensor product Hilbert spaces and their subsets $\mathcal{S}_n$ of rank-n tensors. We then introduce the definition of a projection on the set $\mathcal{S}_1$, which is valid for inner products making the set $\mathcal{S}_1$ weakly closed in V. We prove that this property is satisfied for the classical inner product constructed by tensorization of inner products on individual Hilbert spaces V_i. In section 3, we introduce the definition of a progressive separated representation $z_n \in \mathcal{S}_n$ of an element $z \in V$, based on successive rank-one projections. We prove its convergence in theorem 14, which constitutes a generalization of the Eckart-Young theorem. In section 4, we apply this theorem for proving the convergence of a progressive Proper Generalized Decomposition for a class of linear symmetric elliptic problems in abstract form. In section 5, we finally prove the convergence of a minimal residual progressive Proper Generalized Decomposition for a particular class of linear non-symmetric problems, which uses a minimal residual (least-square) formulation of the problem.

2. Tensor product sums and tensor rank-1 projection

2.1. Tensor product sums on tensor product Hilbert spaces

Let $V = \bigotimes_{i=1}^d V_i$ be a tensor product Hilbert space where V_i, for $i = 1, 2, \ldots, d$, are separable Hilbert spaces. We denote by $(\cdot, \cdot)$ and $\| \cdot \|$ a general inner product on V and its associated norm. We introduce norms $\| \cdot \|_i$ and associated inner products $(\cdot, \cdot)_i$ on V_i, for $i = 1, 2, \ldots, d$. These norms and inner products define a particular norm on V, denoted $\| \cdot \|_V$, defined by

$$\| \otimes_{i=1}^d v_i \|_V = \prod_{i=1}^d \|v_i\|_i,$$

for all $(v_1, v_2, \ldots, v_d) \in \mathbf{V}$, where $\mathbf{V}$ is the product space $V_1 \times \cdots \times V_d$. The associated inner product $(\cdot, \cdot)_V$ is defined by

$$\left(\otimes_{i=1}^d u_i, \otimes_{i=1}^d v_i\right)_V = \prod_{i=1}^d (u_i, v_i)_i,$$

Recall that V, endowed with inner product $(\cdot,\cdot)_V$, is in fact constructed by taking the completion under this inner product.

Now, we introduce the set of V of vectors that can be written as a sum of tensor-rank 1 elements. For each $n \in \mathbb{N}$, we define the set of rank-n tensors

$$\mathcal{S}_n = \{u \in V : \text{rank}_{\otimes} u \leq n\},$$

introduced in [8] in the following way. Given $u \in V$ we say that $u \in \mathcal{S}_1$ if $u = u_1 \otimes u_2 \otimes \cdots \otimes u_d$, where $u_i \in V_i$, for $i = 1, \ldots, d$. For $n \geq 2$ we define inductively $\mathcal{S}_n = \mathcal{S}_{n-1} + \mathcal{S}_1$, that is,

$$\mathcal{S}_n = \left\{ u \in V : u = \sum_{i=1}^{k} u^{(i)},\, u^{(i)} \in \mathcal{S}_1 \text{ for } 1 \leq i \leq k \leq n \right\}.$$

Note that $\mathcal{S}_n \subset \mathcal{S}_{n+1}$ for all $n \geq 1$. We will say for $u \in V$ that $\text{rank}_{\otimes} u = n$ if and only if $u \in \mathcal{S}_n \setminus \mathcal{S}_{n-1}$.

We first consider the following important property of the set $\mathcal{S}_1$ and inner product $\|\cdot\|_V$.

Lemma 1. *$\mathcal{S}_1$ is weakly closed in $(V, \|\cdot\|_V)$.*

Proof. Since

$$\otimes_{i=1}^d v_i = v_1 \left(\prod_{i=2}^{d} \|v_i\|_i \right) \otimes \left(\otimes_{i=2}^d \frac{v_i}{\|v_i\|_i} \right),$$

we may assume, without loss of generality, that $\|v_i\|_i = 1$ for $i = 2, \ldots, d$. Now, assume that the sequence $\{\otimes_{i=1}^d v_i^n\}_{n=1}^{\infty} \subset \mathcal{S}_1$ converges weakly to $v \in V$ in the $\|\cdot\|_V$-norm. It implies that $\{\bigotimes_{i=1}^d v_i^n\}_{n=1}^{\infty}$ is a bounded sequence in the $\|\cdot\|_V$-norm. Moreover, since for $i = 2, \ldots, d$, the sequence $\{v_i^n\}_{n=1}^{\infty}$ is bounded in the $\|\cdot\|_i$-norm, there exists a subsequence $\{v_i^{n_k}\}_{k=1}^{\infty}$ that converges weakly to $v_i^* \in V_i$. Since $\|\bigotimes_{i=1}^d v_i^{n_k}\|_V = \|v_1^{n_k}\|_1$, then $\{v_1^{n_k}\}_{n=1}^{\infty}$ is also bounded in the $\|\cdot\|_1$-norm. In consequence, there exists a further subsequence $\{v_1^{n_k}\}_{k=1}^{\infty}$ that converges weakly to $v_1^* \in V_1$. Clearly, $\{\otimes_{i=1}^d v_i^{n_k}\}_{k=1}^{\infty}$ converges weakly to $\otimes_{i=1}^d v_i^*$ and by the uniqueness of the limit, we obtain that $v \in \mathcal{S}_1$. This proves the lemma. ∎

Since equivalent norms induce the same weak topology on V, we have the following corollary.

Corollary 2. *If the norm $\|\cdot\|$ on V is equivalent to the norm $\|\cdot\|_V$, then $\mathcal{S}_1$ is weakly closed in $(V, \|\cdot\|)$.*

Corollary 3. *If the V_i are finite-dimensional vectors spaces, then $\mathcal{S}_1$ is weakly closed in $(V, \|\cdot\|)$ whatever the norm $\|\cdot\|$.*

2.2. A characterization of a tensor rank-one projection

Now we want to characterize a projection on $\mathcal{S}_1$, called a tensor rank-one projection, with respect to a given inner product $(\cdot,\cdot)$ on V, with associated norm $\|\cdot\|$. We make the following assumption on the inner product.

Assumption 4. *We suppose that inner product $(\cdot,\cdot)$, with associated norm $\|\cdot\|$, is such that $\mathcal{S}_1$ is weakly closed in $(V, \|\cdot\|)$.*

Let us recall that by Corollary 2, the particular norm $\|\cdot\|_V$ verifies Assumption 4.

Definition 5. *A tensor rank-one projection with respect to inner product $(\cdot,\cdot)$, with associated norm $\|\cdot\|$ verifying Assumption 4, is a map $\Pi : z \in V \mapsto \Pi(z) \subset \mathcal{S}_1$ defined by*

$$\Pi(z) = \arg\min_{v\in\mathcal{S}_1} \|z - v\|^2 \tag{3}$$

The following Lemma 6 proves that Assumption 4 is a sufficient condition on the inner product $(\cdot,\cdot)$ for the map Π to be well defined.

Lemma 6. *Under Assumption 4, for each $z \in V$, there exists $v^* \in \mathcal{S}_1$ such that*

$$\|z - v^*\|^2 = \min_{v\in\mathcal{S}_1} \|z - v\|^2$$

Proof. We have

$$\begin{aligned}
\min_{v\in\mathcal{S}_1} \|z - v\|^2 &= \min_{\lambda\in\mathbb{R}, w\in\mathcal{S}_1:\|w\|=1} \|z - \lambda w\|^2 & (4)\\
&= \min_{\lambda\in\mathbb{R}, w\in\mathcal{S}_1:\|w\|=1} \|z\|^2 - \lambda(w,z) + \lambda^2 & (5)\\
&= \min_{w\in\mathcal{S}_1:\|w\|=1} \|z\|^2 - (z,w)^2 & (6)\\
&= \|z\|^2 - \max_{w\in\mathcal{S}_1:\|w\|=1} (z,w)^2 & (7)\\
&= \|z\|^2 - \left(\max_{w\in\mathcal{S}_1:\|w\|=1} (z,w)\right)^2 & (8)
\end{aligned}$$

Since $\mathcal{S}_1$ is a weakly closed set, then the set $\{w \in \mathcal{S}_1 : \|w\| \leq 1\}$ is weakly compact. The existence of minimizers v^* then follows from the existence of maximizers w^* of the linear functional $w \to (z,w)$ on a weakly compact set. To end the proof we need to show that $\|w^*\| = 1$. Assume that $\|w^*\| < 1$ then it follows that

$$(z, w^*) \geq \lambda\,(z, w^*)$$

for all $\lambda \leq 1/\|w^*\|$. In particular, for $\lambda = 1/\|w^*\|$ we obtain $\|w^*\| \geq 1$, a contradiction. ■

We now introduce a generalization of the concept of dominant singular value and dominant singular vectors for an element in a tensor product space.

Definition 7. *The dominant singular value $\sigma(z) \geq 0$ of an element $z \in V$ and the associated set of dominant singular vectors $\mathcal{V}(z)$ are respectively defined by*

$$\sigma(z) = \max_{w \in \mathcal{S}_1 : \|w\|=1} (z, w), \tag{9}$$

and

$$\mathcal{V}(z) = \{w \in \mathcal{S}_1 : \|w\| = 1 \ \ \textit{and} \ \ \sigma(z) = (z, w)\}. \tag{10}$$

The tensor rank-one projector Π can be written

$$\Pi(z) = \sigma(z)\mathcal{V}(z) \tag{11}$$

which means that for $v^* \in \Pi(z)$, there exists $w^* \in \mathcal{V}(z)$ such that $v^* = \sigma(z)w^*$. Let us note that for a given z, $\Pi(z)$ is a multi-valuated map if singular value $\sigma(z)$ is associated with multiple singular vectors. We now introduce other characterization and properties of projector Π.

Theorem 8. *Let $z \in V$. Then the following statements are equivalent*

(a) $v^* \in \Pi(z)$.

(b) $v^* \in \mathcal{S}_1$ *satisfies*

$$\mathcal{E}_z(v^*) = \min_{v \in \mathcal{S}_1} \mathcal{E}_z(v). \tag{12}$$

where the map $\mathcal{E}_z$ is defined as

$$\mathcal{E}_z(v) = \frac{1}{2}\|v\|^2 - (z, v).$$

Moreover,

$$\mathcal{E}_z(v^*) = -\frac{1}{2}\sigma(z)^2 = -\frac{1}{2}\|v^*\|^2, \tag{13}$$

$$\|z - v^*\|^2 = \|z\|^2 - \sigma(z)^2 = \|z\|^2 - \|v^*\|^2, \tag{14}$$

and

$$(z - v^*, v^*) = 0. \tag{15}$$

Proof. Since

$$\mathcal{E}_z(v) \quad = \quad \frac{1}{2}(v, v) - (z, v) = \frac{1}{2}\|z - v\|^2 - \frac{1}{2}\|z\|^2.$$

This implies that the minimization problem (12) is equivalent to

$$\min_{v \in \mathcal{S}_1} \|z - v\|^2, \tag{16}$$

and

$$\min_{v \in \mathcal{S}_1} \mathcal{E}_z(v) \quad = \quad \frac{1}{2}\min_{v \in \mathcal{S}_1} \|z - v\|^2 - \frac{1}{2}\|z\|^2. \tag{17}$$

If $z=0$ then $v^*=0$ and the theorem clearly holds. Now, assume that $z\neq 0$. From (17) and (7) we deduce

$$\min_{v\in\mathcal{S}_1}\mathcal{E}_z(v) = -\frac{1}{2}\max_{w\in\mathcal{S}_1:\,\|w\|=1}(z,w)^2. \tag{18}$$

Thus, $v^*\in\mathcal{S}_1$ solves (12) if and only if $v^*=\sigma(z)w^*$ for some $w^*\in\mathcal{V}(z)$. This follows the first statement of the theorem. To prove the second one, from (18) follows (13) and by using (17) we obtain (14). Finally, from (13) we have that

$$(v^*,v^*)-(z,v^*)=0,$$

and this follows (15). ■

Now, we briefly discuss the particular case $d=2$ and prove that the definition of $\sigma(z)$ in definition (7) is closely related with the classical definition of the dominant singular value of the singular value decomposition of an element $z\in V=V_1\otimes V_2$. By using the Riesz representation theorem, we introduce the following definition. For each $z\in V$ and $w_1\in V_1$ (respectively, $w_2\in V_2$) there exists a unique $\{z,w_1\}_2\in V_2$ (respectively, $\{z,w_2\}_1\in V_1$) such that

$$(z,w_1\otimes w_2)_V = (\{z,w_1\}_2,w_2)_2 \tag{19}$$

for all $w_2\in V_2$, (respectively,

$$(z,w_1\otimes w_2)_V = (\{z,w_2\}_1,w_1)_1 \tag{20}$$

for all $w_1\in V_1$). Observe that since $(u_1\otimes u_2,w_1\otimes w_2)_V=(u_1,w_1)_1(u_2,w_2)_2$ then $\{u_1\otimes u_2,w_1\}_2=(u_1,w_1)_1u_2$ and $\{u_1\otimes u_2,w_2\}_1=(u_2,w_2)_1u_1$. Recall the classical definition of the dominant eigenvalue of a symmetric positive definite operator $A:V_2\to V_2$ as

$$\sigma_1=\max_{\|w\|_2=1}(w,w)_A^{1/2}.$$

The next proposition provide us a classical interpretation of $\sigma(z)$ in the case $d=2$.

Proposition 9. *If $V=V_1\otimes V_2$ and if $(\cdot,\cdot)=(\cdot,\cdot)_V$, which is built from inner products $(\cdot,\cdot)_1$ and $(\cdot,\cdot)_2$ on V_1 and V_2. Then*

$$\sigma(z)=\max_{\|w_2\|_2=1}(\{z,w_2\}_1,\{z,w_2\}_1)_1^{1/2}=\max_{\|w_1\|_1=1}(\{z,w_1\}_2,\{z,w_1\}_2)_2^{1/2}. \tag{21}$$

Proof. We have

$$\sigma(z)=\max_{w\in\mathcal{S}_1;\|w\|=1}(z,w)_V=\max_{\substack{w_1\,\in\,V_1;\,\|w_1\|_1\,=\,1\\ w_2\,\in\,V_2;\,\|w_2\|_2\,=\,1}}(z,w_1\otimes w_2)_V$$

because $\|u_1\otimes u_2\|_V=\|u_1\|_1\|u_2\|_2=1$ and we can write for all $\lambda\in\mathbb{R}\setminus\{0\}$, $u_1=\lambda w_1$ with $\|w_1\|_1=1$ and $u_2=\lambda^{-1}w_2$ with $\|w_2\|_2=1$. Now, let us consider the problem

$$\max_{w_2\in V_2;\|w_2\|_2=1}(z,w_1\otimes w_2)_V=\max_{w_2\in V_2;\|w_2\|_2=1}(\{z,w_1\}_2,w_2)_2 \tag{22}$$

To solve it, we consider the Lagrangian function

$$\mathcal{L}(w_2,\lambda) = (\{z,w_1\}_2, w_2)_2 - \frac{\lambda}{2}\left((w_2,w_2)_2 - 1\right).$$

Since

$$D_{w_2}\mathcal{L}(w_2,\lambda) = (\{z,w_1\}_2,\cdot)_2 - \lambda(w_2,\cdot)_2,$$

the maximum is attained at

$$w_2 = \lambda^{-1}\{z,w_1\}_2.$$

By using that $\|w_2\|_2 = 1$ we obtain $\lambda = \|\{z,w_1\}_2\|_2$. Therefore

$$\sigma(z) = \max_{w_1\in V_1;\|w_1\|_1=1} (\{z,w_1\}_2, \{z,w_1\}_2)_2^{1/2} \tag{23}$$

which is closely related with the classical characterization of the dominant singular value of z. Let us note that in the same way, we could also prove that

$$\sigma(z) = \max_{w_2\in V_2;\|w_2\|_2=1} (\{z,w_2\}_1, \{z,w_2\}_1)_1^{1/2}. \tag{24}$$

■

3. A generalization of the Eckart-Young theorem

Now, we introduce an extension of Eckart-Young theorem, which can be viewed as a generalization of multidimensional singular value decomposition with respect to inner products not necessarily built by tensorization of inner products. We introduce an inner product $(\cdot,\cdot)$ and associated norm $\|\cdot\|$ satisfying Assumption 4. We denote by Π the associated tensor rank-one projector, defined by (3) (or (11)).

Definition 10 (Progressive separated representation of an element in V). *For a given $z \in V$, we define the sequence $\{z_n\}_{n\geqslant 0}$, with $z_n \in \mathcal{S}_n$, as follows: $z_0 = 0$ and for $n \geqslant 1$,*

$$z_n = \sum_{i=1}^{n} z^{(i)}, \quad z^{(i)} \in \Pi(z - z_{i-1}) \tag{25}$$

or equivalently

$$z_n = \sum_{i=1}^{n} \sigma_i w^{(i)}, \quad \sigma_i = \sigma(z - z_{i-1}), \quad w^{(i)} \in \mathcal{V}(z - z_{i-1}) \tag{26}$$

z_n is called an optimal rank-n progressive separated representation of z with respect to the norm $\|\cdot\|$.

We introduce the following definition of the *progressive rank*[2].

Definition 11 (Progressive rank). *We define the progressive rank of an element $z \in V$, denoted by $\operatorname{rank}_\sigma(z)$, as follows:*

$$\operatorname{rank}_\sigma(z) = \min\{n : \sigma(z - z_n) = 0\} \tag{27}$$

where z_n is the progressive separated representation of z, defined in definition 10, where by convention $\min(\emptyset) = \infty$.

Before we state the Generalized Eckart-Young theorem we recall the classical one that is equivalent to the existence of the Singular Value Decomposition.

Theorem 12 (Eckart-Young theorem). *Let $V = \mathbb{R}^n \otimes \mathbb{R}^m$ and let be $\|\cdot\|_F$ the Frobenious norm on V. For each $z \in V$ and $1 \leq n \leq \operatorname{rank} z$, there exists $z_n = \sum_{i=1}^n \sigma_i\, v_i \otimes w_i$ a (nonunique) minimizer of*

$$\min_{w \in \mathcal{S}_n} \|z - w\|_F, \tag{28}$$

where $\sigma_i > 0$, $\|v_i \otimes w_i\|_F = 1$ for $1 \leq i \leq n$, and such that

$$\|z - \sum_{i=1}^n \sigma_i\, v_i \otimes w_i\|_F^2 = \|z\|_F^2 - \sum_{i=1}^n \sigma_i^2 = \sum_{j=n+1}^{\operatorname{rank} z} \sigma_j^2,$$

holds. Here $\operatorname{rank} z$ denotes the matrix rank of $z \in V$.

In this theorem the tensor product over the matrix space $V = \mathbb{R}^n \otimes \mathbb{R}^m$ is defined by $u \otimes v = u \cdot v^T$, where v^T denotes the transpose of vector v. Then, it is not difficult to see that the Frobenius norm $\|z\|_F^2 = \sum_{i=1}^n \sum_{j=1}^m z_{i,j}^2$ is a crossnorm on $\mathbb{R}^n \otimes \mathbb{R}^m$.

Remark 13. *Unfortunately, in [8], it has been proved that tensors of order 3 or higher can fail to have best rank-n approximation, that is, (28) is ill-posed for tensors of order 3 or higher. In consequence, only rank-one approximations are available.*

Now we state the Generalized Eckart-Young theorem.

Theorem 14 (Generalized Eckart-Young theorem). *For $z \in V$, the sequence $\{z_n = \sum_{i=1}^n \sigma_i w^{(i)}\}_{n \geqslant 0}$ constructed in definition 10 verifies:*

$$z = \lim_{n \to \infty} z_n = z_{\operatorname{rank}_\sigma(z)} = \sum_{i=1}^{\operatorname{rank}_\sigma(z)} \sigma_i w^{(i)}$$

and

$$\|z - z_n\|^2 = \|z\|^2 - \sum_{i=1}^n \sigma_i^2 = \sum_{i=n+1}^{\operatorname{rank}_\sigma(z)} \sigma_i^2.$$

[2]Note that in general, the progressive rank $\operatorname{rank}_\sigma$ of an element $z \in V$ is different from the optimal rank $\operatorname{rank}_\otimes(z)$.

Proof. Let $e_{n-1} = z - z_{n-1}$, for $n \geq 1$, where by convention $z_0 = 0$. We have $w^{(n)} = \bigotimes_{i=1}^{d} w_i^n \in \mathcal{V}(e_{n-1})$ and $\sigma_n = \sigma(z - z_{n-1}) = \sigma(e_{n-1})$. We let $z^{(n)} = \sigma_n w^{(n)} \in \mathcal{S}_1$. Let us first note that it holds for $1 \leq n \leq \mathrm{rank}_\sigma(z)$ that $z^{(n)} \neq 0$ since for such n, $\sigma(z - z_{n-1}) > 0$ by definition of the progressive rank. We have

$$\begin{aligned}
\|e_n\|^2 &= \|e_{n-1} - z^{(n)}\|^2 && (29)\\
&= \|e_{n-1}\|^2 - \|z^{(n)}\|^2 \quad \text{(by using (14))} && (30)\\
&= \|e_{n-1}\|^2 - \sigma_n^2 && (31)
\end{aligned}$$

Thus $\{\|e_n\|\}_{n=0}^{\mathrm{rank}_\sigma(z)}$ is a strictly decreasing sequence of non-negative real numbers.

We first assume that $\mathrm{rank}_\sigma(z) = r < \infty$. Then, $\sigma_r = \sigma(z - z_r) = 0$ and $z^{(r+1)} = 0$ since

$$\|z - z_r - z^{(r+1)}\|^2 = \|z - z_r\|^2 - \sigma_r^2 = \|z - z_r\|^2$$

We have

$$\|z - z_r\|^2 = \min_{v \in \mathcal{S}_1} \|z - z_r - v\|^2 \leq \|z - z_r - \lambda v\|^2$$

for all $\lambda \in \mathbb{R}$ and $v \in \mathcal{S}_1$. This implies that

$$\left(z - z_r, \left(\otimes_{i=1}^{d} v_i\right)\right) = 0$$

for all $(v_1, \ldots, v_d) \in \mathbf{V}$. Thus $z - z_r = 0$ and the first statement of theorem follows.

On the other hand, we assume that $\mathrm{rank}_\sigma(z) = \infty$. Then $\{\|e_n\|\}_{n=0}^{\infty}$ is a strictly decreasing sequence of non-negative real numbers, and there exists

$$\lim_{n \to \infty} \|e_n\| = \lim_{n \to \infty} \|z - z_n\| = R \geq 0.$$

Proceeding from (31) and using that $e_0 = z$, we obtain

$$\|e_n\|^2 = \|z\|^2 - \sum_{k=1}^{n} \sigma_k^2. \tag{32}$$

In consequence, $\sum_{k=1}^{\infty} \sigma_k^2$ is a convergent series and $\lim_{n\to\infty} \sigma_n^2 = 0$. Thus, we obtain also

$$\lim_{n \to \infty} \sigma_n = \lim_{n \to \infty} \|z^{(n)}\| = 0. \tag{33}$$

For all $n \geq 1$ and $(w_1, \ldots, w_d) \in \mathbf{V}$ with $\left\|\left(\otimes_{i=1}^{d} w_i\right)\right\| = 1$, we have

$$\left(e_{n-1}, \left(\otimes_{i=1}^{d} w_i\right)\right)^2 \leq \max_{w \in \mathcal{S}_1 :\, \|w\|=1} \left(e_{n-1}, w\right)^2 = \sigma_n^2 \tag{34}$$

and then

$$\lim_{n \to \infty} \left(e_{n-1}, \left(\otimes_{i=1}^{d} w_i\right)\right)^2 = 0 \tag{35}$$

Assume that $\{e_n\}_{n=0}^{\infty}$ is convergent in the $\|\cdot\|$-norm to some $e^* \in V$. Since the sequence is also weakly convergent to e^*, we obtain from (35) that

$$\left(e^*, \left(\otimes_{i=1}^d w_i\right)\right) = 0$$

for all $(w_1, \ldots, w_d) \in \mathbf{V}$ with $\left\|\left(\otimes_{i=1}^d w_i\right)\right\| = 1$. Thus, $e^* = 0$. To conclude the proof we only need to show that $\{e_n\}_{n=1}^{\infty}$ is a Cauchy sequence in V in the $\|\cdot\|$-norm. The following Lemmas will be useful.

Lemma 15. *For each $n, m \geq 1$, it follows that*

$$\left|\left(e_{m-1}, z^{(n)}\right)\right| \leq \sigma_m \sigma_n$$

Proof. We have

$$\left|\left(e_{m-1}, z^{(n)}\right)\right| = \left|\left(e_{m-1}, \sigma_n w^{(n)}\right)\right| = \left|\left(e_{m-1}, w^{(n)}\right)\right| \sigma_n \leq \sigma_m \sigma_n$$

where we have used

$$\sigma_m = \left(e_{m-1}, w^{(m)}\right) = \max_{w \in \mathcal{S}_1 : \|w\|=1} (e_{m-1}, w) \geq \left(e_{m-1}, w^{(n)}\right),$$

■

Lemma 16. *For every $\varepsilon > 0$ and every $N \in \mathbb{N}$ there exists $\tau \geq N$ such that*

$$\sigma_\tau \sum_{k=1}^{\tau} \sigma_k \leq \varepsilon. \tag{36}$$

Proof. Since $\sum_{j=1}^{\infty} \sigma_j^2 < \infty$, for a given $\varepsilon > 0$ and $N \in \mathbb{N}$, we choose $n \geq N$ such that

$$\sum_{j=n+1}^{\infty} \sigma_j^2 \leq \varepsilon/2$$

Since $\lim_{j\to\infty} \sigma_j = 0$, we construct $\tau : \mathbb{N} \longrightarrow \mathbb{N}$ defined inductively by $\tau(1) = 1$ and for all $k \geq 1$,

$$\tau(k+1) = \min_{j > \tau(k)} \left\{\sigma_j \leq \sigma_{\tau(k)}\right\},$$

such that τ is strictly increasing and $\lim_{k\to\infty} \tau(k) = \infty$. Observe that for all $k \geq 1$ and j satisfying $\tau(k) \leq j < \tau(k+1)$, it follows that

$$\sigma_{\tau(k+1)} \leq \sigma_{\tau(k)} \leq \sigma_j.$$

Thus, for all $1 \leq j < \tau(k+1)$, we have

$$\sigma_{\tau(k+1)} \leq \sigma_j$$

Now, since $\lim_{k\to\infty} \sigma_{\tau(k)} = 0$, we can choose $\tau = \tau(k+1) > n$ large enough satisfying

$$\sigma_\tau \sum_{j=1}^{n} \sigma_j \leq \varepsilon/2.$$

Then

$$\begin{aligned}
\sigma_\tau \sum_{j=1}^{\tau} \sigma_j &= \sigma_\tau \sum_{j=1}^{n} \sigma_j + \sigma_\tau \sum_{j=n+1}^{\tau} \sigma_j \quad \le \quad \varepsilon/2 + \sigma_\tau \sum_{j=n+1}^{\tau} \sigma_j \\
&\le \varepsilon/2 + \sum_{j=n+1}^{\tau} \sigma_j^2 \quad \le \quad \varepsilon/2 + \sum_{j=n+1}^{\infty} \sigma_j^2 \\
&\le \varepsilon
\end{aligned}$$

This proves the lemma. ∎

Lemma 17. *For all $M > N > 0$, it follows that*

$$\|e_{N-1} - e_{M-1}\|^2 \quad \le \quad \|e_{N-1}\|^2 - \|e_{M-1}\|^2 + 2\,\sigma_M \sum_{k=1}^{M} \sigma_k$$

Proof. We have

$$\begin{aligned}
\|e_{N-1} - e_{M-1}\|^2 &= \|e_{N-1}\|^2 + \|e_{M-1}\|^2 - 2\,(e_{M-1}, e_{N-1}) \\
&= \|e_{N-1}\|^2 + \|e_{M-1}\|^2 - 2\left(e_{M-1}, e_{M-1} + \sum_{k=N}^{M-1} z^{(k)}\right) \\
&= \|e_{N-1}\|^2 - \|e_{M-1}\|^2 - 2\sum_{k=N}^{M-1}\left(e_{M-1}, z^{(k)}\right) \\
&\le \|e_{N-1}\|^2 - \|e_{M-1}\|^2 + 2\,\sigma_M \sum_{k=N}^{M-1} \sigma_k \quad \text{(by using Lemma 15)} \\
&\le \|e_{N-1}\|^2 - \|e_{M-1}\|^2 + 2\,\sigma_M \sum_{k=1}^{M} \sigma_k \quad \text{(by adding positive terms.)}
\end{aligned}$$

This ends the proof of lemma. ∎

Since the limit of $\|e_n\|^2$ goes to R^2 as $n \to \infty$, and it is a decreasing sequence, for a given $\varepsilon > 0$ there exists $k_\varepsilon > 0$ such that

$$R^2 \le \|e_{m-1}\|^2 \le R^2 + \varepsilon^2/2$$

for all $m > k_\varepsilon$. Now, we assume that $m > k_\varepsilon$. From Lemma 16, for each $m + p$ there exists $\tau > m + p$ such that

$$\sigma_\tau \sum_{k=1}^{\tau} \sigma_k \le \varepsilon^2/4.$$

Now, we would to estimate

$$\|e_{m-1} - e_{m+p-1}\| \le \|e_{m-1} - e_{\tau-1}\| + \|e_{\tau-1} - e_{m+p-1}\|.$$

By using Lemma 17 with $M = \tau$ and $N = m$ and $m + p$, we obtain that

$$\|e_{m-1} - e_{\tau-1}\|^2 \quad \leq \quad R^2 + \varepsilon^2/2 - R^2 + \varepsilon^2/2 = \varepsilon^2,$$

and

$$\|e_{m+p-1} - e_{\tau-1}\|^2 \quad \leq \quad R^2 + \varepsilon^2/2 - R^2 + \varepsilon^2/2 = \varepsilon^2,$$

respectively. In consequence $\{e_n\}_{n=0}^{\infty}$ is a Cauchy sequence in the $\|\cdot\|$-norm and it converges to 0. ■

4. Proper Generalized Decomposition of the solution of a class of linear symmetric elliptic problem

4.1. Formulation of the problem

We consider the following variational problem, defined on the a tensor product Hilbert space $(V, \|\cdot\|_V)$:

$$u \in V, \quad \mathcal{A}(u, v) = L(v) \quad \forall v \in V \tag{37}$$

where $\mathcal{A}(\cdot, \cdot) : V \times V \longrightarrow \mathbb{R}$ is a continuous, symmetric, $V-$elliptic bilinear form, *i.e.* such that for all $u, v \in V$,

$$|\mathcal{A}(u, v)| \leq M\|u\|_V\|v\|_V, \tag{38}$$

$$\mathcal{A}(u, v) = \mathcal{A}(v, u), \tag{39}$$

$$\mathcal{A}(v, v) \geq \alpha\|v\|_V^2 \tag{40}$$

for constants $M > 0$ and $\alpha > 0$.

4.2. Problem in operator form

We introduce the operator $A : V \longrightarrow V$ associated with $\mathcal{A}$, and defined by

$$\mathcal{A}(u, v) = (Au, v)_V \tag{41}$$

for all $u, v \in V$. We also introduce the element $l \in V$ associated with L and defined by

$$L(v) = (l, v)_V \tag{42}$$

for all $v \in V$.. The existence of A and l is ensured by the Riesz representation theorem. Problem (37) can be rewritten in an operator form:

$$Au = l \tag{43}$$

From the assumptions on the bilinear form $\mathcal{A}(\cdot, \cdot)$, we know that A is bounded, self-adjoint, and positive definite, *i.e* for all $u, v \in V$,

$$\|Av\|_V \leq M\|v\|_V,$$

$$(Au, v)_V = (u, Av)_V$$

$$(Av, v)_V \geq \alpha\|v\|_V^2$$

As usual, we will denote by $(\cdot,\cdot)_A$ the inner product induced by the operator A, where for all $u, v \in V$

$$(u, v)_A = (Au, v)_V = (u, Av)_V,$$

We denote by $\|u\|_A = (u, u)_A^{1/2}$ the associated norm. Note that if $A = I$ the identity operator, then $\|\cdot\|_A = \|\cdot\|_V$.

4.3. Rank-one projector based on the A-norm

From properties of operator A, the norm $\|\cdot\|_A$ is equivalent to $\|\cdot\|_V$. Therefore, by Corollary 2, the set $\mathcal{S}_1$ is weakly closed in $(V, \|\cdot\|_A)$ and then, $\|\cdot\|_A$ verifies assumption 4. For a given $z \in V$, we use definition 5 and 7 with $(\cdot,\cdot) = (\cdot,\cdot)_A$ in order to define the rank-one projector $\Pi_A(z)$, the singular value $\sigma_A(z)$ and the set of singular vectors $\mathcal{V}_A(z)$.

4.4. Proper Generalized Decomposition

The progressive Proper Generalized Decomposition (PGD) of the solution $u = A^{-1}l$ of problem (37) is defined as the optimal progressive separated representation defined in Definition 10, associated with projector $\Pi = \Pi_A$. The rank-n progessive PGD is then defined as

$$u_n = \sum_{i=1}^{n} u^{(i)}, \quad u^{(i)} \in \Pi_A(u - u_{i-1}) \tag{44}$$

From properties of the A-norm, the generalized Eckard-Young Theorem 14 ensures the convergence of this sequence.

Remark 18. *Let us note that the proposed progressive PGD is the simplest definition of PGD. Other definitions of PGD have been proposed, which have better convergence properties [19, 20].*

5. Minimal Residual Proper Generalized Decomposition

5.1. Formulation of the problem

We consider the following problem:

$$u \in V, \quad \mathcal{A}(u, v) = L(v) \quad \forall v \in V \tag{45}$$

where $\mathcal{A}$ and L are continuous bilinear and linear forms on V respectively. By Riesz representation, we associate the operator $A : V \to V$ and vector $l \in V$ to bilinear form $\mathcal{A}$ and linear form L, respectively defined by equations (41) and (42). The continuity of $\mathcal{A}$ implies that A is bounded, *i.e.*

$$\exists M > 0 \quad \text{such that} \quad \|Av\|_V \leq M\|v\|_V \tag{46}$$

We further assume the following property on A: for all $v \in V$,

$$\exists c > 0 \quad \text{such that} \quad \|Av\|_V \geq c\|v\|_V \tag{47}$$

5.2. Least-square formulation

We introduce a least-square formulation of problem (45):

$$u \in V, \quad \tilde{\mathcal{A}}(u,v) = \tilde{L}(v) \quad \forall v \in V \tag{48}$$

with

$$\tilde{\mathcal{A}}(u,v) = (A(u), A(v))_D \tag{49}$$

$$\tilde{L}(v) = (l, A(v))_D \tag{50}$$

where $D : V \to V$ is a symmetric continuous and V-elliptic operator which defines an inner product $(\cdot,\cdot)_D$ on V. Bilinear form $\tilde{\mathcal{A}}$ is associated with operator $\tilde{A} = A^* D A$, where A^* is the adjoint operator of A. From properties of A and D, $\tilde{A} : V \to V$ is symmetric continuous and V-elliptic. It then defines an inner product on V, denoted $(\cdot,\cdot)_{\tilde{A}}$, with associated norm $\|\cdot\|_{\tilde{A}}$ which is equivalent to the norm $\|\cdot\|_V$. Formulation (48) is equivalent to the following minimal residual formulation:

$$u = \arg\min_{v\in V} \frac{1}{2}\|A(v) - l\|_D^2 = \arg\min_{v\in V} \frac{1}{2}\|v - A^{-1}l\|_{\tilde{A}}^2 \tag{51}$$

5.3. Progressive Minimal Residual Proper Generalized Decomposition

Since $\|\cdot\|_{\tilde{A}}$ is equivalent to $\|\cdot\|_V$ on V, $\mathcal{S}_1$ is weakly closed in $(V, \|\cdot\|_{\tilde{A}})$, by Corollary 2. We can then define a tensor rank-one projection $\Pi_{\tilde{A}}$ associated with $\tilde{A}$, as long as the dominant singular value $\sigma_{\tilde{A}}(z)$ and the associated set of dominant singular vectors $\mathcal{V}_{\tilde{A}}(z)$, for each $z \in \mathcal{V}$.

The minimal residual progressive Proper Generalized Decomposition (PGD) of the solution $u = A^{-1}l$ of problem (45) is defined as the optimal progressive separated representation defined in Definition 10, associated with projector $\Pi = \Pi_{\tilde{A}}$. A rank-n minimal residual progessive PGD is defined as

$$u_n = \sum_{i=1}^{n} u^{(i)}, \quad u^{(i)} \in \Pi_{\tilde{A}}(u - u_{i-1})$$

From properties of the $\tilde{A}$-norm, the generalized Eckard-Young Theorem 14 ensures the convergence of this sequence.

Remark 19. *The convergence of the minimal residual PGD strongly depends on the choice of the D-norm. Choosing for D the identity operator on V, corresponding to $(\cdot,\cdot)_D = (\cdot,\cdot)$, usually leads to very poor convergence properties (although it is very convenient from a computational point of view). Choosing a "good" D is a critical problem. A compromise must be made between good convergence properties of u_n and computational issues related to the construction of u_n.*

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