An entropy formula for a class of circle maps†

by

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Abstract. In this note we give a simple formula to compute the topological entropy of a certain class of degree one circle maps which depends only on the "kneading pair" of the map under consideration. The class of maps we consider generalizes the one-parameter family of maps whose bifurcations were studied by Hockett and Holmes in [3].

Rsum. Dans cette note on donne une formule simple pour calculer l'entrope topologique d'une application du cercle en lui-mme. Cette formule seulement depend de la "paire de petrissage" de l'application. La classe d'applications qu'on considre est une gnralisation de la famille un paramtre donc les bifurcations ont t tudis par Hocket et Holmes [3].

† The authors have been partially supported by the DGICYT grant number PB90–0695.

Version franaise abrge

Dans cette note on utilise la thorie des itinraires symboliques pour applications du cercle en lui-mme dveloppe par Alsed et Maosas [1] pour obtenir une formule pour l'entropie topologique des applications du cercle dans une classe particulire. Cette classe est une gnralisation de la famille d'applications tudie par Hocket et Holmes dans [3].

On note par $\mathcal{L}$ l'ensemble d'applications continues $F : \mathbf{R} \to \mathbf{R}$ vrifiant $F(x+1) = F(x) + 1$ (c'est a dire $\mathcal{L}$ est l'ensemble de relvements des applications continues du cercle en lui-mme de degr un). On dit qu'une application $F \in \mathcal{M}$ si:

(A) $F \in \mathcal{L}$.

(B) Il existe $c_F \in (0,1)$ tel que F est strictement croissante dans l'intervalle $[0, c_F]$ et strictement dcroissante dans $[c_F, 1]$.

(C) Il existe un intervalle ferm $A_F \subset (0,1)$ tel que $c_F \in \text{Int}(A_F)$ et $F(A_F) \subset A_F + m$ pour $m \in \mathbf{Z}$.

Soit $F \in \mathcal{L}$ un relvement d'une application f du cercle en lui- mme. On dira que l'entrope topologique de F, dnote par $h(F)$, est l'entropie topologique de f (voir [2] pour la dfinition d'entrope topologique de f).

Le rsultat principal de la note est le suivant.

Thorme 1 *Soit $F \in \mathcal{M}$. Alors il existe K_F et P_F, deux polynmes dpendant seulement de $\{F^n(0)\}_{n=0}^{\infty}$ et $\{F^n(c_F)\}_{n=0}^{\infty}$ respectivement, tels que $h(F) = \log(\min\{\alpha_{K_F}, \alpha_{P_F}\})^{-1}$ o α_{K_F} et α_{P_F} sont, respectivement, les plus petites racines de K_F et P_F dans l'intervalle $(0,1)$.*

Les corollaires suivantes donnent des conditions dans lesquelles la formule pour calculer l'entrope topologique d'applications de $\mathcal{M}$ est encore plus simple.

Corollaire 2 *Si la longueur de l'intervalle de rotation de $F \in \mathcal{M}$ est plus grand que $1/2$, alors $h(F) = \log \alpha_{P_F}^{-1}$.*

Corollaire 3 *Soit $F \in \mathcal{M}$ tel que $E(F(c_F)) - E(F(0)) \geq 2$. Alors $h(F) = \log \alpha_{P_F}^{-1}$ (o $E(\cdot)$ dnote la fonction partie entire).*

Finalement on donne une formule pour calculer l'entrope topologique de la famille d'applications tudie par Hocket et Holmes dans [3]. Soit $[\mu_0, \mu_1]$ un intervalle ferme de la droite rel et soit $F_\mu = F(\mu, .) : [\mu_0, \mu_1] \times \mathbf{R} \longrightarrow \mathbf{R}$ une famille d'applications qui dpend continment du paramtre μ et qui, pour tout $\mu \in [\mu_0, \mu_1]$, satisfait les conditions suivantes:

(a) $F_\mu \in \mathcal{M} \cap \mathcal{C}^1(\mathbf{R}, \mathbf{R})$.

(b) L'application $(F_\mu - m)|_{A_{F_\mu}}$ a le point fixe rpulsif $\min A_{F_\mu}$ et un point fixe attractif $w_\mu \in A_{F_\mu}$.

(c) On a $a \in (0, c_F)$ et $b \in (c_F, 1)$ tels que $F_\mu(b) = F_\mu(\min A_{F_\mu}) = F_\mu(a) + 1$ et $a + 1 > F_\mu(0) > b$.

Alors on a

Corollaire 4 *Pour la famille d'applications prcdante on a $h(F_\mu) = \log \alpha_{P_{F_\mu}}^{-1}$.*

1 Introduction

In [3] Hockett and Holmes describe certain bifurcations of a continuous one-parameter family of degree one circle maps in terms of the relation between the parameter and the rotation interval of these maps. To carry on their study they use the natural extension of the “Kneading Theory” of Milnor and Thurston [5] to the family of maps they consider. This extension is based in the use of an “ad hoc” coding. In order to maintain small the number of symbols of this coding (and, therefore, to maintain the difficulty of the computations at a reasonable level) the authors have to impose a restriction on the “height” of the maps under consideration (see Section 2 for a precise definition of “height”).

The purpose of this paper is to obtain a simple formula for the topological entropy of the maps from the family considered by Hockett and Holmes in [3]. To do this, instead of working in their framework, we shall use the coding introduced by Alsed and Maosas in [1] together with the appropriate extension of the “Kneading Theory” to this coding. The advantage of this approach is that it allows us to work with circle maps of degree one of arbitrary ”height” without increasing too much the difficulty of the computations. In fact, we shall be able to find a simple entropy formula for a much wider class of maps that the one considered by Hockett and Holmes [3]. This formula depends in a simple way on the “kneading pair” of the map under consideration (see again Section 2 for a precise definition of a “kneading pair”).

Now we are going to define the class $\mathcal{M}$ of maps we shall consider. As usual, given a continuous map of the circle into itself we shall work with its lifting rather than with the map itself. Thus we shall consider the class $\mathcal{L}$ of continuous maps $F : \mathbf{R} \to \mathbf{R}$ such that $F(x+1) = F(x) + 1$ (that is, $\mathcal{L}$ is the class of all liftings of continuous circle maps of degree one). The we shall say that $F \in \mathcal{M}$ if:

(A) $F \in \mathcal{L}$.

(B) There exists $c_F \in (0,1)$ such that F is strictly increasing in $[0, c_F]$ and strictly decreasing in $[c_F, 1]$.

(C) There exists a closed interval A_F of length at most 1 such that $c_F \in \text{Int}(A_F)$ and $F(A_F) \subset A_F + m$ for some $m \in \mathbf{Z}$.

We note that each map from $\mathcal{L}$ having a minimum in $[0,1]$ is conjugated by a translation to a map from $\mathcal{L}$ having the minimum at 0. Therefore, the fact that in (B) we fix that F has a minimum in 0 is not restrictive. On the other hand, it is not difficult to see that for $F \in \mathcal{M},\ 0 \notin A_F$. Moreover, if $1 \in A_F$ then the map $F - m$ restricted to an appropriate interval of length 1 is a bimodal map of the interval. Since an entropy formula for such maps has already been obtained in [6] we can replace (C) by the following stronger condition:

(D) There exists a closed interval $A_F \subset (0,1)$ such that $c_F \in \text{Int}(A_F)$ and $F(A_F) \subset A_F + m$ for some $m \in \mathbf{Z}$.

We note that then $(F-m)|_{A_F}$ is unimodal. The next section will be devoted to introduce the coding we are going to use together with the appropriate “Kneading Theory” and, in Section 3, we shall state the main result of the paper, which gives the entropy formula we are looking for.

2 Kneading sequences and topological entropy for maps in $\mathcal{M}$

In this section, we are going to outline the extension of the kneading theory of Milnor and Thurston [5] to the class $\mathcal{M}$. These techniques have been used already by Alsed and Maosas in

[1] to obtain lower bounds of the topological entropy depending on the rotation interval for the class of maps satisfying conditions (A) and (B).

We start by introducing some notation. In what follows we shall denote by $E(.)$ the integer part function. For $F \in \mathcal{M}$ we define *the height of* F, denoted by p_F, as $E(F(c_F)) - E(F(0))$.

If $A \subset \mathbf{R}$ and $x \in \mathbf{R}$, we shall write $x + A$ or $A + x$ to denote the set $\{x + a : a \in A\}$. Let $F \in \mathcal{M}$ be with height p. Then the points of the set $\Delta(F) = \mathbf{Z} \cup F^{-1}(\mathbf{Z}) \cup c_F + \mathbf{Z}$ will be called the *turning points* of F. We note that if $x \in \Delta(F)$ then $x + \mathbf{Z} \subset \Delta(F)$. Moreover, $\Delta(F) \cap [0, 1]$ can be written as $\{c_0, c_1, c_2, \dots, c_{2p+1}\}$ with $0 = c_0 < c_1 < \dots < c_{p+1} = c_F < \dots < c_{2p+1} = 1$, $F(c_1) = E(F(0)) + 1 = E(F(c_F)) - p + 1$ and $F(c_i) = F(c_{2p+1-i}) = E(F(c_F)) - p + i$ for $i = 2, 3, \dots, p$.

Now we define the notion of address we are going to use. For $x \in \mathbf{R}$ we set $A_F(x) = (s(x), d(x))$, where $d(x) = E(F(x)) - E(x)$ and

$$s(x) = \begin{cases} L & \text{if } x - E(x) < c_F \text{ and } x \notin \Delta(F), \\ R & \text{if } x - E(x) > c_F \text{ and } x \notin \Delta(F), \\ c_i & \text{if } D(x) = c_i. \end{cases}$$

We note that $F|_{[c_{i-1}, c_i]}$ is monotone and $(E \circ F)|_{[c_{i-1}, c_i]}$ is constant for all $i = 1, 2, \dots, 2p+1$. Hence, each point from an interval of the form $(c_{i-1}, c_i) + m$ with $m \in \mathbf{Z}$ has the same address.

Let $A = (s, d) \in \{L, R, c_0, c_1, c_2, \dots, c_{2p+1}\} \times \mathbf{Z}$. We set $\epsilon(L) = 1$, $\epsilon(R) = -1$, and $\epsilon(c_i) = 0$ for all $i = 0, 1, \dots, 2p+1$. We also set $\kappa_0(x) = A_F(x)$ and $\kappa_n(x) = \left[\prod_{i=0}^{n-1} \epsilon(A_F(F^i(x)))\right] A_F(F^n(x))$ for each $n \in \mathbf{N}$. Then the power series $\sum_{n=0}^{\infty} \kappa_n(x) t^n$ will be called the *invariant coordinate of* x and will be denoted by $\kappa_F(x)$ (or simply $\kappa(x)$ when no confusion will be possible). Note that $\kappa(x) = \kappa(x + m)$ for all $m \in \mathbf{Z}$.

Let $\mathcal{V}$ be the set of all pairs of the form (s, d) with $d \in \mathbf{Z}$ and $s \in \{L, R\}$. We note that for $F \in \mathcal{M}$ and for $x \notin \Delta(F)$, $A_F(x) \in \mathcal{V}$.

It is not difficult to show that for each $n \geq 0$ there exists $\delta(n) > 0$ such that $\kappa_n(y)$ takes a constant value, denoted by $\kappa(x^+)$, for all $y \in (x, x + \delta(n))$. Then, for $x \in \mathbf{R}$ we set

$$\kappa(x^+) = \kappa_F(x^+) = \sum_{n=0}^{\infty} \kappa_n(x^+) t^n.$$

In a similar way we define $\kappa(x^-)$. If $F^n(x) \notin \Delta(F)$ for all $n \geq 0$ (that is, $s(F^n(x)) \in \{L, R\}$ for all $n \geq 0$) then $\kappa(x^+) = \kappa(x^-) = \kappa(x)$. As for the invariant coordinate we have that $\kappa(x^+) = \kappa((x + m)^+)$ and $\kappa(x^-) = \kappa((x + m)^-)$ for all $m \in \mathbf{Z}$.

Remark 2.1 For each $\delta > 0$ there exists $\epsilon > 0$ such that for all $x \in (0, \delta)$ there exists $y \in (-\epsilon, 0)$ with the property that $F(x) = F(y)$. Therefore we get that $\kappa_0(0^+) = (L, E(F(0)))$, $\kappa_0(0^-) = (R, E(F(0)) + 1)$ and $\kappa_n(0^+) = -\kappa_n(0^-)$ for all $n > 0$. In a similar way we obtain that $\kappa_0(c_F^+) = (R, E(F(c_F)))$, $\kappa_0(c_F^-) = (L, E(F(c_F)))$ and $\kappa_n(c_F^+) = -\kappa_n(c_F^-)$ for all $n > 0$. □

The sequences $\kappa(0^+)$ and $\kappa(c_F^-)$ play a special role in our study. They will be called the *kneading pair of* F. We note that if one knows the kneading pair of F, in view of Remark 2.1, one can get easily the sequences $\kappa(0^-)$ and $\kappa(c_F^+)$.

By, setting $L < R$ we can define an ordering in $\mathcal{V}$ as follows. Let (s, d) and (t, m) be elements of $\mathcal{V}$ such that $(s, d) \neq (t, m)$. We say that $(s, d) < (t, m)$ if either $s < t$ or $s = t = L$ and $d < m$, or $s = t = R$ and $d > m$. If none of these holds we say that $(s, d) > (t, m)$. The above ordering has the property that if $x, y \notin \Delta(F)$ and $x < y$, then $A_F(x) \leq A_F(y)$.

For a map $F \in \mathcal{M}$, we shall denote by $\mathcal{V}_F$ the set of all addresses of all points of $\mathbf{R} \setminus \Delta(F)$. Note that $\mathcal{V}_F \subset \mathcal{V}$ and $\text{Card}\mathcal{V}_F = 2p + 1$. We also shall write the elements of $\mathcal{V}_F$ as $I_1 < I_2 < \ldots < I_{2p+1}$.

Finally, let $F \in \mathcal{L}$ and assume that F is a lifting of the circle map f. We define the *topological entropy of* F, denoted by $h(F)$ as the topological entropy of f (see [2] for a definition of topological entropy).

3 Topological entropy for maps in $\mathcal{M}$

This section will be devoted to establish the formula for the topological entropy we are looking for. Prior to state the main result of this paper we shall introduce some more notation.

Set $R_F(t) = t[\kappa(0^+) - \kappa(0^-)]$. Since $\kappa(0^+)$ and $\kappa(0^-)$ are formal power series with coefficients in $\mathbf{Z}[[\mathcal{V}_F]]$ so is $R_F(t)$. Hence, $R_F(t)$ can be written as $\sum_{i=1}^{2p_F+1} \phi_i(t) I_i$, where $\phi_i(t) \in \mathbf{Z}[[t]]$ for all $i = 1, 2, \ldots, 2p_F + 1$. Then we also set

$$P_F(t) = -1 + \sum_{i=1}^{p_F} (p_F - i + 1)\phi_i(t) - \sum_{i=p_F+3}^{2p_F+1} (i - p_F - 2)\phi_i(t).$$

Remark 3.1 The series $P_F(t)$ can be computed directly from $\kappa(0^+)$. To see this we note that, in a similar way as we did for $R_F(t)$, we can write $\kappa(0^+)$ as $\sum_{i=1}^{2p+1} \tilde{\phi}_i(t) I_i$ with $\tilde{\phi}_i(t) \in \mathbf{Z}[[t]]$ for all $i = 1, 2, \ldots, 2p_F + 1$. Then, by Remark 2.1, we have that $R_F(t) = tI_1 - tI_{2p+1} + 2t[\kappa(0^+) - I_1] = -tI_1 - tI_{2p+1} + 2t\kappa(0^+)$. Hence, $\phi_1(t) = -t + 2t\tilde{\phi}_1(t)$, $\phi_{2p+1}(t) = -t + 2t\tilde{\phi}_{2p+1}(t)$ and $\phi_i(t) = \tilde{\phi}_i(t)$ for $i = 2, 3, \ldots, 2p$. □

From the definition of $\mathcal{M}$ (see (D)) we have that $\kappa(c_F^+)$ and $\kappa(c_F^-)$ are formal power series with coefficients in $\mathbf{Z}[[I_{p+1}, I_{p+2}]]$. Therefore, $\kappa(c_F^+) - \kappa(c_F^-)$ can be written as $K_F(t)I_{p+1} + \widetilde{K}_F(t)I_{p+2}$ with $K_F(t), \widetilde{K}_F(t) \in \mathbf{Z}[[t]]$.

Remark 3.2 The series $K_F(t)$ can be computed directly from $\kappa(c_F^-)$. Indeed, if $\kappa(c_F^-) = \pi_1(t)I_{p+1} + \pi_2(t)I_{p+2}$ with $\pi_1(t), \pi_2(t) \in \mathbf{Z}[[t]]$ then, by Remark 2.1, we have that $\kappa(c_F^+) = (1 - \pi_1(t))I_{p+1} + (1 - \pi_2(t))I_{p+2}$. Hence $K_F(t) = 1 - 2\pi_1(t)$. □

If $K_F(t)$ vanishes in $(0, 1)$ we shall denote by α_{K_F} the smallest zero of $K_F(t)$ in $(0, 1)$. Otherwise we set $\alpha_{K_F} = 1$. In a similar way we define α_{P_F} by using $P_F(t)$ instead of $K_F(t)$.

The following theorem is the main result of this paper and gives the formula we are looking for.

Theorem 3.3 *For* $F \in \mathcal{M}$ *we have* $h(F) = \log(\min\{\alpha_{K_F}, \alpha_{P_F}\})^{-1}$.

We note that, in view of Remarks 3.1 and 3.2, the numbers α_{K_F} and α_{P_F} can be computed solely from the knowledge of $\kappa(0^+)$ and $\kappa(c_F^-)$. Therefore, Theorem 3.3 gives a formula for the topological entropy of a map from $\mathcal{M}$ *depending only on the kneading pair of the map under consideration.*

In view of Condition (D), for each $F \in \mathcal{M}$ we get that $F|_{A_F}$ is unimodal. Therefore, $\alpha_{K_F}^{-1} \leq 2$ (see for instance [4]). Hence, whenever $\alpha_{P_F}^{-1} \geq 2$ we shall have $h(F) = \log \alpha_{P_F}^{-1}$. Next we shall obtain sufficient conditions to assure the validity of this last formula.

Corollary 3.4 *If the length of the rotation interval of* $F \in \mathcal{M}$ *is strictly larger that* $1/2$ *then* $h(F) = \log \alpha_{P_F}^{-1}$.

If for $F \in \mathcal{M}$ we have that $p_F \geq 2$ then the rotation interval of F has length larger than or equal to 1. Thus, from the above corollary, we obtain

Corollary 3.5 *Let $F \in \mathcal{M}$. If $p_F \geq 2$ then $h(F) = \log \alpha_{P_F}^{-1}$.*

In the case of the family considered by Hockett and Holmes [3] it turns out that $\alpha_{K_F} = 1$ and, hence, the same formula for the topological entropy holds. To see this let us define precisely the family of maps they considered. Let $[\mu_0, \mu_1]$ be a closed proper interval of the real line and let $F_\mu = F(\mu, .) : [\mu_0, \mu_1] \times \mathbf{R} \longrightarrow \mathbf{R}$ be a family which depends continuously on μ and such that, for each $\mu \in [\mu_0, \mu_1]$, it satisfies the following conditions:

(a) $F_\mu \in \mathcal{M} \cap \mathcal{C}^1(\mathbf{R}, \mathbf{R})$.

(b) The map $(F_\mu - m)|_{A_{F_\mu}}$ has $\min A_{F_\mu}$ as a repulsive fixed point and an attractive fixed point $w_\mu \in A_{F_\mu}$.

(c) There exist $a \in (0, c_F)$ and $b \in (c_F, 1)$ such that $F_\mu(b) = F_\mu(\min A_F) = F_\mu(a) + 1$ and $a + 1 > F_\mu(0) > b$.

Then we have

Corollary 3.6 *For the above family of maps we have $h(F_\mu) = \log \alpha_{P_{F_\mu}}^{-1}$.*

References

[1] Alsedà L., Mañosas F., *Kneading theory and rotation interval for a class of circle maps of degree one*, Nonlinearity, **3**, 413-452, 1990.

[2] Denker M., Grillenberger C., Sigmund K., *Ergodic Theory in compact spaces*, Lecture Notes in Mathematics **527**, 1976.

[3] Hockett K. Holmes P., *Bifurcations to badly ordered orbits in one parameter families of circle maps, or Angels fallen from the Devil's staircase*, Proc. Amer. Math. Soc. **102** (1988), 1031–1051.

[4] Misiurewicz M. Szlenk W., *Entropy of piecewise monotone mappings*, Stud. Math., **67**, 45-63, 1980.

[5] Milnor J. Thurston P., *On iterated maps on the interval*, Dynamical systems, 465–563, Lecture Notes in Math. **1342**, Springer, Berlin, 1988.

[6] Mumbr P. *Periodic structure and topological entropy of bimodal maps (in Catalan)*, Ph. D. Thesis, Universitat Autnoma de Barcelona.

Titre en Franais: **Une formule pour l'entrope topologique pour une classe d'aplications du cercle**

RUBRIQUE: Systèmes Dynamiques

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